Tensor Analysis for Engineers

Transformations-Mathematics-Applications

SECOND EDITION

M. TABATABAIAN

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Second Edition

TENSOR ANALYSIS FOR ENGINEERS

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Mehrzad Tabatabaian, PhD, PEng

MERCURY LEARNING AND INFORMATION
Dulles, Virginia
Boston, Massachusetts
New Delhi

Publisher: David Pallai
Mercury Learning and Information
22841 Quicksilver Drive
Dulles, VA 20166
info@merclearning.com
www.merclearning.com
(800) 232-0223

M. Tabatabaian. *Tensor Analysis for Engineers: Transformations-Mathematics-Applications, Second Edition.*
ISBN: 978-1-68392-601-6

Library of Congress Control Number: 2020946631

202122321 This book is printed on acid-free paper in the United States of America.

To my teachers and mentors
for their invaluable transfer of knowledge and direction.

CONTENTS

PREFACE

This is the second edition of the published textbook: *Tensor Analysis for Engineers*. In this edition, we expand the content on the rigid body rotation and Cartesian tensors by including Euler angles and quaternions methods. In addition to the rotation matrix method, presented in the first edition and included in this edition, we collect all three methods in this volume of the textbook. In this edition, the quaternions and their algebraic calculation rules are presented. We also discuss the active and passive rotations and present several worked-out examples using the Euler angles and quaternions methods applications and their interrelations. The problem of gimbal lock is also analyzed and presented with detailed worked out examples. Additional references have been included in the second edition.

In engineering and science, physical quantities are often represented by mathematical functions, namely *tensors*. Examples include temperature, pressure, force, mechanical stress, electric/magnetic fields, velocity, enthalpy, entropy, etc. In turn, tensors are categorized based on their rank, i.e. rank zero, one, and so forth. The so-called scalar quantities (e.g., temperature) are tensors of rank zero. Likewise, velocity and force are tensors of rank one and mechanical stress and gradient of velocity are tensors of rank two. In Euclidean space, which could be of dimension $N = 3, 4, \ldots$, we can define several coordinate systems for our calculation and measurement of physical quantities. For example, in a 3D space, we can have Cartesian, cylindrical, and spherical coordinate systems. In general, we prefer defining a coordinate system whose coordinate surfaces (where one of the coordinate variables is invariant or remains constant) match to the physical problem geometry at hand. This enables us to easily define the boundary conditions of the physical problem to the related governing equations, written in terms of the selected coordinate system. This action requires transformation of the tensor quantities and their related derivatives (e.g., gradient, curl, divergence) from Cartesian to the selected coordinate system or vice versa. The topic of *tensor analysis* (also referred to as "tensor calculus" or "Ricci's calculus" since originally developed by Ricci, 1835–1925, [1], [2]), is mainly engaged with the definition of tensor-like quantities and their transformation among coordinate systems and others. The topic provides a set of mathematical tools which enables users to perform transformation and calculations of tensors for any well-defined coordinate systems in a systematic way—it is a "machine." The merit of tensor analysis is to provide a systematic mathematical formulation to derive the general form of the governing equations for arbitrary coordinate systems.

In this book, we aim to provide engineers and applied scientists the tools and techniques of tensor analysis for applications in practical problem solving and analysis activities. The geometry is limited to the Euclidean space/geometry, where the Pythagorean Theorem applies, with well-defined Cartesian coordinate systems as the reference. We discuss quantities defined in curvilinear coordinate systems, like cylindrical, spherical, parabolic, etc., and present several examples and coordinates sketches with related calculations. In addition, we listed several worked-out examples for helping the readers with mastering the topics provided in the prior sections. A list of exercises is provided for further practice for readers.

Mehrzad Tabatabaian, PhD, PEng
Vancouver, BC
September 30, 2020

ABOUT THE AUTHOR

Dr. Mehrzad Tabatabaian is a faculty member at the Mechanical Engineering Department, School of Energy at BCIT. He has several years of teaching and industry experiences. In addition to teaching courses in mechanical engineering, he performs research on renewable energy systems and modelling. Dr. Tabatabaian is currently Chair of the BCIT School of Energy Research Committee. He has published several papers in scientific journals and conferences, and he has written textbooks on multiphysics and turbulent flow modelling, advanced thermodynamics, tensor analysis, and direct energy conversion. He holds several registered patents in the energy field.

Recently, Dr. Tabatabaian volunteered to help establish the Energy Efficiency and Renewable Energy Division (EERED), a new division at Engineers and Geoscientists British Columbia (EGBC).

Mehrzad Tabatabaian received his BEng from Sharif University of Technology and advanced degrees from McGill University (MEng and PhD). He has been an active academic, professor, and engineer in leading alternative energy, oil, and gas industries. Mehrzad has also a Leadership Certificate from the University of Alberta and holds an EGBC P.Eng. license.

CHAPTER 1

INTRODUCTION

Physical quantities can be represented mathematically by *tensors*. In further sections of this book we will define tensors more rigorously; however, for the introduction we will use this definition. An example of a tensor-like quantity is the temperature in a room (which could be a function of space and time) expressed as a scalar, a tensor of rank zero. Wind velocity is another example, which can be defined when we know both its magnitude and speed—a scalar quantity—and its direction. We define velocity as a vector or a tensor of rank one. Scalars and vectors are familiar quantities to us and we encounter them in our daily life. However, there are quantities, or tensors of rank two, three, or higher that are normally dealt with in technical engineering computations. Examples include mechanical stress in a continuum, like the wall of a pressure vessel—a tensor of rank two—the modulus of elasticity or viscosity in a fluid—tensors of rank four—and so on.

Engineers and scientists calculate and analyze tensor quantities, including their derivatives, using the laws of physics, mostly in the form of governing equations related to the phenomena. These laws must be expressed in an objective form as governing equations, and not subjected to the coordinate system considered. For example, the amount of internal stress in a continuum should not depend on what coordinate system is used for calculations. Sometimes tensors and their involved derivatives in a study must be transformed from one coordinate system to another. Therefore, to satisfy these technical/engineering needs, a mathematical "machine" is required to perform these operations accurately and systematically between arbitrary coordinate systems. Furthermore, the communication of technical computations requires precise definitions for tensors to guarantee a reliable level of standardization, i.e., identifying true tensor quantities from apparently tensor-like or non-tensor ones. This machine is called *tensor analysis* [2], [3], [4], [5], [6], [7].

The subject of tensor analysis has two major parts: a) definitions and properties of tensors including their calculus, and b) rules of transformation of tensor quantities among different coordinate systems. For example, consider again the wind velocity vector. By using tensor analysis, we can show that this quantity is a true tensor and transform it from a Cartesian coordinate system to a spherical one, for example. A major outcome of tensor analysis is having general relations for gradient-like operations in arbitrary coordinate systems, including gradient, curl, divergence, Laplacian, etc. which appear in many governing equations in engineering and science. Using tensor notation and definitions, we can write theses governing equations in explicit coordinate-independent forms.

1.1 INDEX NOTATION—THE EINSTEIN SUMMATION CONVENTION

Writing expressions containing tensors could become cumbersome, especially when higher ranked tensors and higher dimensional space are involved. For example, we usually use hatted arrow symbol for vectors (like $\vec{A}$) and bold-font symbols for second rank tensors (like $\boldsymbol{A}$). But this approach is very limited for expressing, for example, the modulus of elasticity, a 4$^{\text{th}}$ ranked tensor. Another limitation shows up when writing the components of a tensor in N-dimension space. For example, in 3D we write vector $\vec{A}$ in a Cartesian coordinate system as $\vec{A}=\sum_{1}^{3}A_i\vec{E}_i = A_1\vec{E}_1 + A_2\vec{E}_2 + A_3\vec{E}_3$, where A_i is the component and $\vec{E}_i$ the unit vector. Taking this approach for N-dimension space and carrying the summation symbol (i.e., Σ) is cumbersome and seems unnecessary.

The Einstein summation convention allows us to dispose of the summation symbol, if we carry summation operation for repeated indices in product-type expressions (unless otherwise specified). Using this approach, we can write $\vec{A}=A_i\vec{E}_i$ and a tensor of rank 4, for example, as $\boldsymbol{T}=T_{ijkl}\vec{E}_i\vec{E}_j\vec{E}_k\vec{E}_l$, with all indices range for 1 to N. In practice, however, we even ignore writing the unit vectors and just use the component representing the original tensor; hence this method is also referred to as *component* or *index* notation, or $\vec{A}\equiv A_i$ and $\boldsymbol{T}\equiv T_{ijkl}$. In addition, we represent a tensor's rank by the number of free (i.e., not repeated) indices. We use these definitions and conventions throughout this book.

CHAPTER 2

COORDINATE SYSTEMS DEFINITION

For measuring and calculating physical quantities associated with geometrical points in space we require coordinate systems for reference. These systems (for example, in a 3D space) are composed of surfaces that mutually intersect to specify a geometrical point. For reference, an ideal system of coordinates called a Cartesian system is defined, in Euclidean space, such that it composes of three flat planes. These planes are considered by default to be mutually perpendicular to each other to form an *orthogonal* Cartesian system. In general, the orthogonality is not required to form a coordinate system—this kind of coordinate system is an *oblique* or slanted system. A Cartesian coordinate system, although ideal, is central to engineering and scientific calculations, since it is used as the reference compared to other coordinate systems. It also has properties, as will be shown in further sections, that enable us to calculate the values of tensors [3].

For practical purposes, sometimes it is more convenient to consider curved planes instead of flat ones in a coordinate system. For example, for cases when a cylindrical or a spherical water or an oil tank is involved, we prefer that all or some of coordinate surfaces match the geometry of the tank. For a cylindrical coordinate system, we consider the Cartesian system again but replace one of its planes with a cylinder, whereas for a spherical system we replace two flat planes with a sphere and a cone.

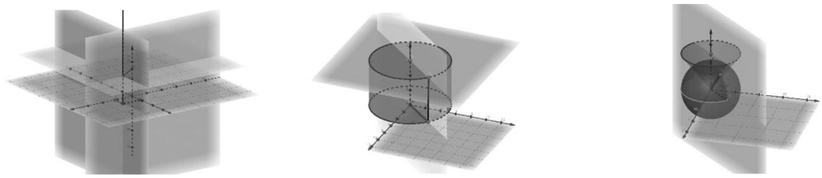

FIGURE 2.1 Sketches of Cartesian, Cylindrical, and Spherical coordinate systems.

Many other coordinate systems are defined/used in practice such as parabolic, bi-spherical, etc. Figure 2.1 shows some examples of common coordinate systems. Animations of the coordinates shown in Figure 2.1 are provided in the accompanying files.

For organizing the future usage of symbols for arbitrary coordinate systems we define coordinate variables with superscripted letters, for reasons that will be explained in further sections. For Cartesian systems we use y^i and for other arbitrary systems we use x^i—note that i is merely an index, not a raised power. For example, in a 3D space we have $y^1 \equiv X$, $y^2 \equiv Y$, and $y^3 \equiv Z$ where (X, Y, Z) are common notations for Cartesian coordinates. In general, for N dimensional systems we have

$$\begin{cases} y^i \equiv \left(y^1, y^2, y^3, \cdots, y^N\right) & \textit{Cartesian coordinates} \\ x^i \equiv \left(x^1, x^2, x^3, \cdots, x^N\right) & \textit{arbitrary coordinates} \end{cases} \tag{2.1}$$

In this book, we limit the geometry to that known as Euclidean geometry [2], [4], [5]. Therefore, curved space geometry (i.e., Riemann geometry), space-time, and discussions of General Relativity are not included. However, curvilinear coordinate systems and oblique non-orthogonal systems are covered, where applicable, and defined with reference to Euclidean geometry/space.

CHAPTER 3

BASIS VECTORS AND SCALE FACTORS

For the measurement of quantities we need to define metrics or scales in whatever coordinate system we use. These scales are, usually, vectors defined at a point (such as the origin) and are tangent to the coordinate surfaces at that point. These vectors are called *basis vectors.* For Cartesian system y^i we define basis vector $\vec{E}_i$ (see Figure 3.1). Note that subscript i (i.e., $i = 1, 2, \cdots, N$) is merely an index and the reason why we used it as a subscript for basis vectors will be explained in a further section. Now, considering an incremental vector $\overrightarrow{ds}$, from point P to neighboring point P', we define the direction of the basis vector as moving from P to P' or $\vec{E}_i = y^i\big|_P \rightarrow y^i + dy^i\big|_{P'}$. For example, $\vec{E}_1 = y^1\big|_P \rightarrow y^1 + dy^1\big|_{P'}$. Now we define the magnitude of $\vec{E}_i$ such that the magnitude of distance from P to P' is given as

$$ds(i) = \left|dy^i \vec{E}_i\right| \tag{3.1}$$

Also, we can get the distance $\overline{PP'} = ds(i) = dy^i$ in a Cartesian coordinate system. Therefore, the basis vectors magnitude is unity, or $\left|\vec{E}_i\right| = 1$. In other words, the basis vectors in Cartesian system have unit lengths and are well-known unit vectors (usually represented by $\vec{i}, \vec{j}, \vec{k}$). This is more than a trivial result and as shown in further sections, enables us to calculate/measure quantities in other coordinate systems with reference to the Cartesian system. The directed distance vector $\overrightarrow{ds}$ is given as

$$\overrightarrow{ds} = \sum_{i=1}^{N} dy^i \vec{E}_i = dy^i \vec{E}_i \tag{3.2}$$

The magnitude of this vector is the square root of the dot-product of $\overrightarrow{ds}$ with itself,

$$\left|\overrightarrow{ds}\right| = \sqrt{\overrightarrow{ds}\cdot\overrightarrow{ds}} = \sqrt{dy^i dy^j \vec{E}_i \cdot \vec{E}_j} \tag{3.3}$$

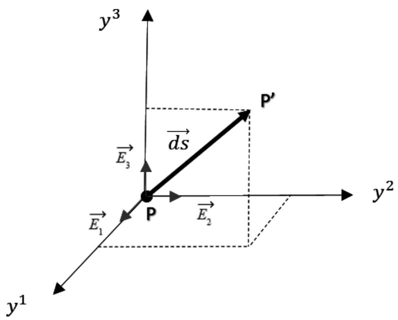

FIGURE 3.1 A Cartesian system with unit vectors and an incremental vector $\overrightarrow{ds}$.

Assuming the Pythagorean theorem holds in Euclidean space we have

$$\left|\overrightarrow{ds}\right| = \sqrt{dy^i dy^i} = \sqrt{\sum_i \left(dy^i\right)^2} \tag{3.4}$$

From Equations 3.3 and 3.4, we can conclude that

$$\vec{E}_i \cdot \vec{E}_j = \begin{cases} 1 \ \ for\, i = j \\ 0 \ for\, i \neq j \end{cases} \tag{3.5}$$

Which also shows that the basis/unit vectors in Cartesian system y^i are mutually perpendicular or the system is orthogonal, as defined.

Now we assume an arbitrary system x^i, which may be neither rectilinear nor orthogonal (see Figure 3.2). In this system the distance between two points—say P and P'—is the same when considering the vector $\overrightarrow{ds}$. In other words, the vector components in system x^i compared to those in system y^i could change along with the basis vector corresponding to system x^i such that the vector itself remains the same, or as an *invariant*. This requirement is simply a statement of independence of physical quantities (and the laws of nature) regardless of the coordinate system we consider for calculation and analysis. Therefore, we have

$$\overrightarrow{ds} = dx^i \vec{e}_i = dy^i \vec{E}_i \tag{3.6}$$

where $\vec{e}_i$ is the basis vector corresponding to system x^i. In general the basis vector $\vec{e}_i$ can vary, both in magnitude and/or direction, from point to point in space. Also, the dx^i may be dimensionless, like the angle coordinate in a polar coordinate system.

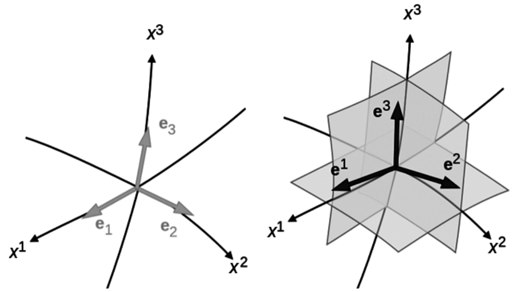

FIGURE 3.2 Sketches of a curvilinear coordinate system and basis vectors[1].

The magnitude of basis vector $\vec{e}_i$ is the scale factor h_i, or

$$h_i = |\vec{e}_i| \tag{3.7}$$

Note that h_i is unity for a Cartesian coordinate system and may be different from unity in general curvilinear systems. We will derive formulae for the calculation of h_i in an arbitrary coordinate system in further sections. The *unit vector* $\vec{e}(i)$ can be defined as

$$\vec{e}(i) = \frac{\vec{e}_i}{h_i} \quad (\text{no summation on } i) \tag{3.8}$$

Obviously, in a Cartesian system the unit vectors and the basis vectors are identical, since h_i is unity.

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CHAPTER 4

CONTRAVARIANT COMPONENTS AND TRANSFORMATIONS

For transformations between systems x^i and Cartesian y^i, we must have the functional relationships between their coordinate variables. For example,

$$x^i = F_i\left(y^1, y^2, \cdots, y^N\right) \tag{4.1}$$

gives N number of functions transforming y^i to x^i system. Inversely [4], we can transform from x^i to y^i system using function G_i given as

$$y^i = G_i\left(x^1, x^2, \cdots, x^N\right) \tag{4.2}$$

For example, for a 2D polar coordinate system, $(r,\theta) \equiv \left(x^1, x^2\right)$ with reference to Cartesian system $(X,Y) \equiv \left(y^1, y^2\right)$ we have

$$\begin{cases} X = r\cos\theta \\ Y = r\sin\theta \end{cases} \equiv \begin{cases} y^1 = x^1\cos\left(x^2\right) \\ y^2 = x^1\sin\left(x^2\right) \end{cases}. \text{ Or inversely,}$$

$$\begin{cases} r = \sqrt{X^2 + Y^2} \\ \theta = \tan^{-1}\left(\dfrac{Y}{X}\right) \end{cases} \equiv \begin{cases} x^1 = \sqrt{\left(y^1\right)^2 + \left(y^2\right)^2} \\ x^2 = \tan^{-1}\left(\dfrac{y^2}{y^1}\right) \end{cases}.$$

Now by taking partial derivatives of G_i (see Equation 4.2) we can relate the differentials dy^i, in a Cartesian system, to dx^i in an arbitrary coordinate system as

$$dy^i = \frac{\partial y^i}{\partial x^j} dx^j \qquad 4.3$$

For example, after expanding Equation 4.3 for $i=1$ we get, $dy^1 = \frac{\partial y^1}{\partial x^1} dx^1 + \frac{\partial y^1}{\partial x^2} dx^2 + \cdots + \frac{\partial y^1}{\partial x^N} dx^N$. The transformation coefficient $\frac{\partial y^i}{\partial x^j}$ can be written in matrix form as $\frac{\partial y^i}{\partial x^j} = \begin{bmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^N} \\ \vdots & \ddots & \vdots \\ \frac{\partial y^N}{\partial x^1} & \cdots & \frac{\partial y^N}{\partial x^N} \end{bmatrix}$. The determinant of the transformation coefficient matrix is defined as the Jacobian of the transformation, or

$$\mathcal{J} = \frac{\partial\left(y^1, y^2, \cdots, y^N\right)}{\partial\left(x^1, x^2, \cdots, x^N\right)} = \begin{vmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^N} \\ \vdots & \ddots & \vdots \\ \frac{\partial y^N}{\partial x^1} & \cdots & \frac{\partial y^N}{\partial x^N} \end{vmatrix} \qquad 4.4$$

The Jacobian can be interpreted as the density of the space. In other words, let's say that we have $\mathcal{J} = 5$ for a given system x^i. This means that we have packed 5 units of Cartesian space into a volume in x^i space through transformation from Cartesian to the given system. The smaller the Jacobian, the smaller the space density would be and vice versa.

Similarly, we can use function F_i (see Equation 4.1) to have

$$dx^i = \frac{\partial x^i}{\partial y^j} dy^j \qquad 4.5$$

It can be shown [4], [2], that the determinant of transformation coefficient $\frac{\partial x^i}{\partial y^j}$ is the inverse of $\mathcal{J}$, or

$$\mathcal{J}^{-1} = \frac{\partial\left(x^1, x^2, \cdots, x^N\right)}{\partial\left(y^1, y^2, \cdots, y^N\right)} = \begin{vmatrix} \frac{\partial x^1}{\partial y^1} & \cdots & \frac{\partial x^1}{\partial y^N} \\ \vdots & \ddots & \vdots \\ \frac{\partial x^N}{\partial y^1} & \cdots & \frac{\partial x^N}{\partial y^N} \end{vmatrix} \quad 4.6$$

Equations 4.3 and 4.5 show a pattern for the transformation of differentials dy^i and dx^i, respectively. That is, the partial derivatives of the corresponding coordinates appear in the numerator of the transformation coefficient. Nevertheless, one can ask: does this pattern maintain for general system-to-system transformation? The short answer is "yes" [7]. Hence, for arbitrary systems x^i and x'^i we have

$$dx^i = \frac{\partial x^i}{\partial x'^j} dx'^j \quad \text{and} \quad dx'^i = \frac{\partial x'^i}{\partial x^j} dx^j \quad 4.7$$

Any quantity, say A^i, that transforms according to Equation 4.7 is defined as a *contravariant* type, with the standard notation of having the index i as a superscript. Therefore, transformation $A^i \rightleftarrows A'^i$ reads

$$A'^i = \frac{\partial x'^i}{\partial x^j} A^j \quad \text{and} \quad A^j = \frac{\partial x^j}{\partial x'^i} A'^i \quad 4.8$$

Obviously, performing the related calculations requires the functional relations between the two systems, i.e., $x^i = func\left(x'^1, x'^2, \cdots, x'^N\right)$ or $x'^i = func\left(x^1, x^2, \cdots, x^N\right)$. This in turn requires having the Cartesian system as a reference for calculating the values of A^j or A'^i, since x^i and x'^i are arbitrary. For example, transforming A^i directly from a spherical to a cylindrical system requires having the functional relations between these coordinates with reference to the Cartesian system as the main reference.

The contravariant component of a vector has a geometrical meaning as well. To show this, we consider a rectilinear non-orthogonal system (x^1, x^2) in 2D, as shown in Figure 4.1. The components of vector $\vec{A}$ can be obtained by drawing parallel lines to the coordinates x^1 and x^2 to find contravariant components A^1 and A^2, respectively.

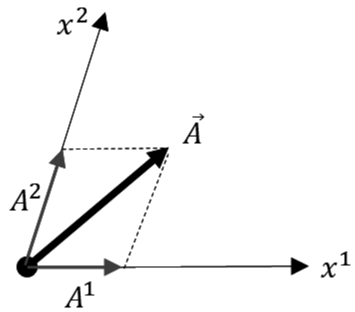


FIGURE 4.1 Contravariant components of a vector in an oblique coordinate system.

With reference to Figure 4.1, we can also find another set of components, A_1 and A_2 of the same vector $\vec{A}$ by drawing perpendicular lines to the coordinates x^1 and x^2. This is shown in Figure 4.2. Obviously, components A_1 and A_2 are different in magnitude from the contravariant components. We define A_1 and A_2 as *covariant* components. In the next section we define the transformation rule for covariant quantities.

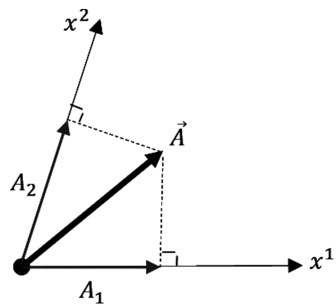


FIGURE 4.2 Covariant components of a vector in an oblique coordinate system.

We can conclude from these definitions that contravariant and covariant components of a vector in a Cartesian system are identical and there is no distinction between them.

CHAPTER 5

COVARIANT COMPONENTS AND TRANSFORMATIONS

We use the standard notation of writing the index i as a subscript for covariant quantities. We consider basis vector $\vec{e}_i$ as a covariant quantity. We can rewrite Equation 3.6, considering two arbitrary systems x^j and x'^k and the fact that $\overrightarrow{ds}$ is coordinate-system independent or invariant, as

$$\overrightarrow{ds} = dx'^k \vec{e}'_k = dx^j \vec{e}_j \qquad 5.1$$

After substituting for dx^j, using Equation 4.7, we get $dx'^k \vec{e}'_k = \frac{\partial x^j}{\partial x'^k} dx'^k \vec{e}_j$ or

$$dx'^k \left(\vec{e}'_k - \frac{\partial x^j}{\partial x'^k} \vec{e}_j \right) = 0 \qquad 5.2$$

Since dx'^k is arbitrary (i.e., the relation is valid for any choice of system x'^k and selections of $dx'^{k=i} \neq 0$ and $dx'^{k \neq i} = 0$, for all values of $i = 1, 2, \cdots, N$) therefore the expression in the bracket must be equal to zero, or we have

$$\vec{e}'_k = \frac{\partial x^j}{\partial x'^k} \vec{e}_j \qquad 5.3$$

All quantities, say A_i, that transform according to Equation 5.3 defined as *covariant* type, with the standard notation of writing the index i as a subscript. Therefore, transformation $A_i \rightleftarrows A'_j$ reads

$$A_i = \frac{\partial x'^j}{\partial x^i} A'_j \text{ and } A'_j = \frac{\partial x^i}{\partial x'^j} A_i \qquad 5.4$$

See Figure 4.2 for the geometrical interpretation of the covariant component of a vector.

CHAPTER 6

PHYSICAL COMPONENTS AND TRANSFORMATIONS

Having defined the contravariant and covariant quantities, like those of the components of a vector, one can ask this question: which one of these two types of components is the actual vector's components in magnitude? The short answer is "none"! However, note that the combination of the contravariant or covariant components and their corresponding basis vectors (contravariant basis vectors will be defined in further section) gives the magnitude of the vector correctly. To find the actual magnitude of the vector, we should find the unit vectors and then the components of the vector corresponding to these unit vectors, or the *physical components*. In other words, the physical components of a vector (or in general, a tensor) are scalars whose physical dimensions and magnitudes are those of the components vectors tangent to the coordinates' lines. Therefore, for vector $\vec{A}$, if we designate $A(i)$ as its physical component and the corresponding unit vector as $\vec{e}(i)$ we can write, then

$$\vec{A} = A^i \vec{e}_i = A(i)\vec{e}(i) \qquad 6.1$$

Now, using Equation 3.8 and substituting for $\vec{e}_i$ we can write $A^i \vec{e}(i) h_i = A(i)\vec{e}(i)$. Therefore,

$$A(i) = A^i h_i, \text{ (no summation on } i) \qquad 6.2$$

Or $A(1) = A^1 h_1$, $A(2) = A^2 h_2$, and so forth. The scale factor h_i is the parameter that turns the contravariant component into the physical component, when multiplied by it. Again, we can conclude that in a Cartesian system the physical and contravariant/covariant components are identical, since $h_i = 1$.

Up to this point we have defined contravariant, covariant, and physical components of a vector along with covariant and physical/unit basis vectors. In principle, we can represent a vector by any combination of its components with the corresponding basis vectors. This requires having the contravariant basis vector definition. We will define this quantity and how it is transformed in a further section. First, we must have the definition of a tensor, as well as a specific tensor called a *metric tensor*.

CHAPTER 7

TENSORS—MIXED AND METRIC

In previous sections, we defined vectors or tensors of rank one. Higher order tensors could be defined based on similar definitions. For example, a tensor of rank two requires more than one free index, like mechanical stress or strain tensors. In general, we have the contravariant components written as A^{ij} for tensor A. We avoid placing double-arrows over the symbol for tensors, for simplicity and generality. Since, in principle, we can have tensors of rank N, designating them with N number of hatted arrows is not a practical exercise. The invariant quantity is written as

$$A = A^{ij}\vec{e}_i\vec{e}_j \qquad 7.1$$

Since A is invariant, its value remains the same regardless of the coordinate system used. We transform A from an arbitrary coordinate system x'^i to another system, say x^i, or $A'^{ij}\vec{e}'_i\vec{e}'_j = A^{km}\vec{e}_k\vec{e}_m$. Now, using Equation 5.3, we transform the covariant basis vectors $\vec{e}_k$ and $\vec{e}_m$ to system x'^i. Therefore, we have $A^{km}\vec{e}_k\vec{e}_m = A^{km}\dfrac{\partial x'^i}{\partial x^k}\dfrac{\partial x'^j}{\partial x^m}\vec{e}'_i\vec{e}'_j$. Using these last two relations, we obtain $\vec{e}'_i\vec{e}'_j\left(A'^{ij} - \dfrac{\partial x'^i}{\partial x^k}\dfrac{\partial x'^j}{\partial x^m}A^{km}\right) = 0$. Therefore, since basis vectors are non-zero, we have

$$A'^{ij} = \frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^m}A^{km} \qquad 7.2$$

Quantities that transform according to Equation 7.2 are defined as doubly contravariant tensors, of rank two. Similarly, we can define mixed tensors, for example of rank two, as

$$A'^{i}_{j} = \frac{\partial x'^{i}}{\partial x^{k}} \frac{\partial x^{m}}{\partial x'^{j}} A^{k}_{m} \tag{7.3}$$

The rule for transformation of tensors is like those given for vectors, i.e., the contravariant component of the tensor transforms like a contravariant vector and the covariant component like a covariant vector. Also, we can expand the definition to a doubly covariant tensor, like

$$A'_{ij} = \frac{\partial x^{k}}{\partial x'^{i}} \frac{\partial x^{m}}{\partial x'^{j}} A_{km} \tag{7.4}$$

Equations 7.2–7.4 are useful relations defining second rank tensors and their transformations. Expanding this rule, we can write the transformation of a N-rank tensor, for example contravariant components, as

$$A'^{ij\cdots N} = \frac{\partial x'^{i}}{\partial x^{k}} \frac{\partial x'^{j}}{\partial x^{m}} \cdots \frac{\partial x'^{N}}{\partial x^{M}} A^{km\cdots M} \tag{7.5}$$

In an arbitrary curvilinear system, the Pythagorean Theorem applies but does not appear in the same form as it does in flat Cartesian systems. This is mainly due to the fact that basis vectors in an arbitrary system change in magnitude and/or direction, from point to point. Therefore, measuring the distance ds between two points on a curved surface, like a sphere, for example, requires scaling the coordinates by multiplying them with some scalar coefficients. These coefficients are the components of a tensor, called a *metric tensor,* associated with the coordinate system selected. Recalling Equation 3.6, we can find the square of the magnitude of differential distance $\overrightarrow{ds}$ by dot-product operation, or $ds^2 = \overrightarrow{ds} \cdot \overrightarrow{ds} = \left(dx^i \vec{e}_i\right) \cdot \left(dx^j \vec{e}_j\right)$. The result is

$$ds^2 = dx^i dx^j \left(\vec{e}_i \cdot \vec{e}_j\right) \tag{7.6}$$

The term in the bracket, on the R.H.S of Equation 7.6, is a scalar (i.e., a tensor of rank zero) resulting from a dot-product operation on the basis vectors. Hence, we can write it as

$$\vec{e}_i \cdot \vec{e}_j = \left|\vec{e}_i\right| \left|\vec{e}_j\right| \cos\alpha = h_i h_j \cos\alpha \tag{7.7}$$

where α is the angle between the tangents to x^i and x^j axes of the coordinate system at the selected point in space. Geometrically speaking, Equation 7.7 is the multiplication of the magnitude of a basis vector with the projection of the other one along the direction of the original one. Therefore, this quantity could be used to identify whether a system (or at least the basis vectors selected) is orthogonal or not. Also, it could be a metric for measuring the distance over a curved surface. It has more properties and applications, which we will encounter in future sections; hence, it is designated by a symbol and a name, i.e., *metric tensor* g_{ij}

$$g_{ij} = \vec{e}_i \cdot \vec{e}_j \tag{7.8}$$

It is easily seen from Equation 7.8 that a metric tensor is symmetric since order in a dot-product is irrelevant (i.e., commutative). Also, if $g_{ij} = 0$ for $i \neq j$ then the coordinates of the system x^i are orthogonal. For example, the metric tensor in a 3D orthogonal system can be represented by a 3×3 matrix containing null off-diagonal elements,

$$g_{ij} = \begin{bmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{bmatrix}, \textit{orthogonal system} \tag{7.9}$$

It would be useful to apply metric tensor definitions and expand Equation 7.6 using Equations 7.7 and 7.9 for an orthogonal 3D system, which gives

$$ds^2 = h_1^2 \left(dx^1\right)^2 + h_2^2 \left(dx^2\right)^2 + h_3^2 \left(dx^3\right)^2 \tag{7.10}$$

For example, in a Cartesian system we recover the familiar form of the Pythagorean theorem (i.e., $ds^2 = (dX)^2 + (dY)^2 + (dZ)^2$), since $g_{11} = g_{22} = g_{33} = \underbrace{\vec{E}_i \cdot \vec{E}_i}_{\textit{no sum on } i} = 1$. Similarly, for a spherical coordinate system (r, φ, θ) we have $h_1^2 = g_{11} = 1$, $h_2^2 = g_{22} = r^2$, and $h_3^2 = g_{33} = r^2 \sin^2 \varphi$ (see Example 8.2 for scale factors). Therefore, we obtain $ds^2 = (dr)^2 + r^2 (d\varphi)^2 + r^2 \sin^2 \varphi (d\theta)^2$.

So far, we have not shown that g_{ij} is a tensor. To do this, we transform the metric tensor to another arbitrary coordinate system, say x'^i. We then can write $g'_{ij} = \vec{e}'_i \cdot \vec{e}'_j = \dfrac{\partial x^k}{\partial x'^i} \dfrac{\partial x^m}{\partial x'^j} \underbrace{\left(\vec{e}_k \cdot \vec{e}_m\right)}_{= g_{km}}$. Therefore, we have $g'_{ij} = \dfrac{\partial x^k}{\partial x'^i} \dfrac{\partial x^m}{\partial x'^j} g_{km}$; that is, when compared to Equation 7.4 we can conclude that g_{km} is a doubly covariant tensor.

As mentioned in the previous sections, to calculate or measure tensor-like quantities we use a Cartesian system as the reference. To evaluate g_{ij} we use Equation 7.8 and transform the covariant basis vector $\vec{e}_i$ to the Cartesian system, y^i. Therefore, $g_{ij} = \vec{e}_i \cdot \vec{e}_j = \frac{\partial y^k}{\partial x^i}\frac{\partial y^m}{\partial x^j}\left(\vec{E}_k \cdot \vec{E}_m\right)$. The quantity $\vec{E}_k \cdot \vec{E}_m$, or the dot-product of the basis/unit vectors in the Cartesian system, is either one or zero, due to orthogonality of the coordinates. Hence, $\vec{E}_k \cdot \vec{E}_m = \begin{cases} 1, \text{ when } k = m \\ 0, \text{ when } k \neq m \end{cases}$. Using this property, we can write

$$g_{ij} = \frac{\partial y^k}{\partial x^i}\frac{\partial y^k}{\partial x^j} \qquad 7.11$$

Using Equation 7.11, we can readily conclude that the metric tensor in a Cartesian system is the familiar Kronecker delta, $\delta_{ij} = \vec{E}_i \cdot \vec{E}_j$. For 3D Cartesian coordinates, we receive a diagonal unity matrix $\delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

It can be shown ([3], [4]) that the determinant of a metric tensor is equal to the square of the Jacobian of the transformation matrix, or

$$g = \left|g_{ij}\right| = \mathcal{J}^2 \qquad 7.12$$

Therefore, in an orthogonal system x_i, having $g_{ij} = 0$ for $i \neq j$, we receive

$$g = g_{11}g_{22}\cdots g_{NN} = h_1^2 h_1^2 \cdots h_N^2, \text{orthogonal system} \qquad 7.13$$

Note that $g_{ii} = \left|\vec{e}_i\right|\left|\vec{e}_i\right| = h_i^2$ (no sum on i) (see Equation 7.7). From Equations 7.12 and 7.13 we can conclude that the Jacobian for an orthogonal system is equal to the products of the scale factors, or

$$\mathcal{J} = h_1 h_2 \cdots h_N \ , \quad \text{orthogonal system} \qquad 7.14$$

For example, in a spherical coordinate system the Jacobian is $\mathcal{J}_{spherical} = h_r h_\varphi h_\theta = r^2 \sin\varphi$ (see Example 8.2 for scale factors).

CHAPTER 8

METRIC TENSOR OPERATION ON TENSOR INDICES

An important and useful application of metric tensor is that it can be used to lower and raise indices of a tensor. Therefore, we can change a contravariant component of a tensor to a covariant one, or vice versa, by multiplying the appropriate metric tensor to the tensor at hand. For example, let's define the covariant component of a vector (i.e., a tensor of rank one) as $A_i = A^j g_{ij}$, where A^j is the given contravariant component. Now, we show that the quantity $A^j g_{ij}$ is transformed like a covariant quantity, according to Equation 5.4. Considering systems x^i and x'^i, we can write $A'^j g'_{ij} = \left(\frac{\partial x'^j}{\partial x^k} A^k\right)\left(\frac{\partial x^m}{\partial x'^i}\frac{\partial x^n}{\partial x'^j} g_{mn}\right) = \left(\frac{\partial x'^j}{\partial x^k}\frac{\partial x^n}{\partial x'^j}\right)\frac{\partial x^m}{\partial x'^i} A^k g_{mn}$. Note that the expression $\frac{\partial x'^j}{\partial x^k}\frac{\partial x^n}{\partial x'^j} = \frac{\partial x^n}{\partial x^k} = \delta^n_k$, where $\delta^n_k = \begin{cases} 1\ , n = k \\ 0, n \neq k \end{cases}$ is a mixed second rank tensor (a Kronecker delta). Therefore, we have $\left(\frac{\partial x'^j}{\partial x^k}\frac{\partial x^n}{\partial x'^j}\right)\frac{\partial x^m}{\partial x'^i} A^k g_{mn} = \frac{\partial x^m}{\partial x'^i}\underbrace{\left(\delta^n_k A^k\right)}_{=A^n} g_{mn} = \frac{\partial x^m}{\partial x'^i}\left(A^n g_{mn}\right)$. The expression in the last bracket is actually A_m, according to our definition. Finally, we have $A'_i = A'^j g'_{ij} = \frac{\partial x^m}{\partial x'^i} A_m$, which clearly shows that the quantity $A'^j g'_{ij}$ is transformed like a covariant component (see Equation 5.4). Therefore, we have, also using the symmetry property of a metric tensor or $g_{ij} = g_{ji}$,

$$A_i = A^j g_{ij} = A^j g_{ji} \tag{8.1}$$

Using Equation 8.1, we can write the expression for a contravariant basis vector, or

$$\vec{e}_i = \vec{e}^{\,j} g_{ij} \qquad 8.2$$

which shows that the metric tensor lowers the contravariant index, while the dummy index is dropped out. This will enable us to write the vector $\vec{A}$ in terms of covariant or contravariant basis vectors, as

$$\vec{A} = A^i \vec{e}_i = A^i \left(g_{ij} \vec{e}^{\,j} \right) = \left(A^i g_{ij} \right) \vec{e}^{\,j} = A_j \vec{e}^{\,j} \qquad 8.3$$

Note that by using physical components and corresponding unit vectors, we can write

$$\vec{A} = A^i \vec{e}_i = A_j \vec{e}^{\,j} = A(k) \vec{e}(k) \qquad 8.4$$

So far, we have defined the doubly covariant metric tensor g_{ij}. Similarly, the doubly contravariant metric tensor is defined $g^{ij} = \vec{e}^{\,i} \cdot \vec{e}^{\,j}$. Therefore, we can write

$$\vec{A} = A^i \vec{e}_i = A_j g^{ij} \vec{e}_i = A_j \vec{e}^{\,j}$$

Also, the combination of the two types of metric tensor gives the mixed one, as

$$g^{ij} g_{jk} = \delta^i_k \qquad 8.5$$

Equation 8.5 can be derived as follow: we have $A^i = g^{ij} A_j = g^{ij} \underbrace{g_{jk} A^k}_{=A_j}$. In this expression, we write $A^i = \delta^i_k A^k$, therefor we have $A^k \left(\delta^i_k - g^{ij} g_{jk} \right) = 0$, but since A^k is arbitrary the expression in the bracket is zero. Hence $g^{ij} g_{jk} = \delta^i_k$.

With reference to Figure 3.2, Figure 4.1, Figure 4.2, and Equation 8.3 we can sketch the vector $\vec{A}$ in terms of its covariant/contravariant components with their corresponding basis vectors, as shown in Figure 8.1 (see Example 15.3). Users may also want to watch a related video at

https://www.youtube.com/watch?v=CliW7kSxxWU.

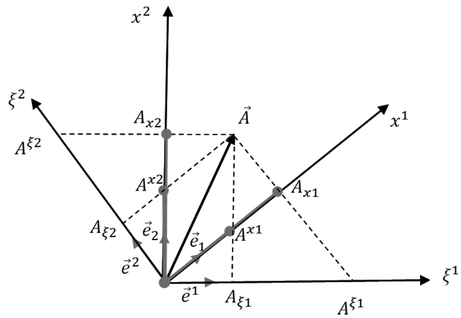


FIGURE 8.1 Sketch for covariant and contravariant components of a vector and basis vectors in a non-orthogonal coordinate system x^i and alternate system ξ^i.

In the following subsections, we present two examples demonstrating related calculations for cylindrical and spherical coordinate systems.

8.1 EXAMPLE: CYLINDRICAL COORDINATE SYSTEMS

Consider a cylindrical polar coordinate system $\left(x^1, x^2, x^3\right) \equiv (r, \theta, z)$ where r is the radial distance, θ the azimuth angle, and z the elevation from $X-Y$ plane, w.r.t. the Cartesian coordinates $\left(y^1, y^2, y^3\right) \equiv (X, Y, Z)$, as shown in Figure 8.2. The functional relations corresponding to cylindrical and Cartesian systems are:

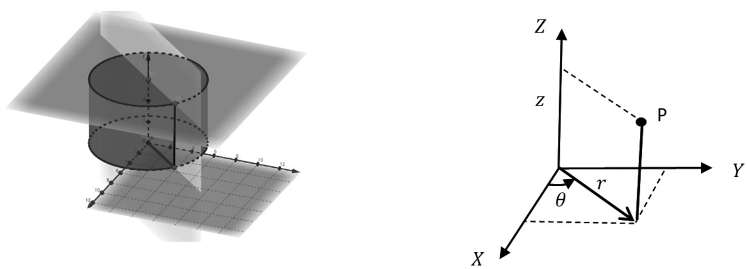


FIGURE 8.2 Cylindrical coordinate system.

$\begin{cases} X = r\cos\theta \\ Y = r\sin\theta \\ Z = z \end{cases} \equiv \begin{cases} y^1 = x^1 \sin x^2 \\ y^2 = x^1 \sin x^2 \\ y^3 = x^3 \end{cases}$. Find the inverse relations (i.e., $x^i = F_i\left(y^1, y^2, \cdots, y^N\right)$, the basis vectors $\vec{e}_i$ and $\vec{e}^{\,i}$ for the cylindrical coordinate system in terms of the Cartesian unit vectors. Also find the scale factors, unit vectors, and metric tensors g_{ij} and g^{ij} and line element magnitude $\left|\overrightarrow{ds}\right|$.

Solution:

The functions F_i can be obtained by solving for X, Y, Z, using the functional relation, given

$$\begin{cases} r = \sqrt{X^2 + Y^2} \\ \theta = \tan^{-1}\left(\dfrac{Y}{X}\right) \\ z = Z \end{cases} \equiv \begin{cases} x^1 = \sqrt{\left(y^1\right)^2 + \left(y^2\right)^2} \\ x^2 = \tan^{-1}\left(\dfrac{y^1}{y^2}\right) \\ x^3 = Y^3 \end{cases}$$

. To find the covariant basis vectors, we use $\vec{e}_i = \dfrac{\partial y^j}{\partial x^i}\vec{E}_j$. Therefore, we receive $\vec{e}_1 = \dfrac{\partial y^1}{\partial x^1}\vec{E}_1 + \dfrac{\partial y^2}{\partial x^1}\vec{E}_2 + \dfrac{\partial y^3}{\partial x^1}\vec{E}_3 = \cos\theta\vec{E}_1 + \sin\theta\,\vec{E}_2$. Similarly, $\vec{e}_2 = \dfrac{\partial y^1}{\partial x^2}\vec{E}_1 + \dfrac{\partial y^2}{\partial x^2}\vec{E}_2 + \dfrac{\partial y^3}{\partial x^2}\vec{E}_3 = -r\sin\theta\vec{E}_1 + r\cos\theta\vec{E}_2$, and $\vec{e}_3 = \dfrac{\partial y^1}{\partial x^3}\vec{E}_1 + \dfrac{\partial y^2}{\partial x^3}\vec{E}_2 + \dfrac{\partial y^3}{\partial x^3}\vec{E}_3 = \vec{E}_3$. In terms of coordinates notation, we have

$$\begin{cases} \vec{e}_r = \cos\theta\vec{E}_1 + \sin\theta\,\vec{E}_2 \\ \vec{e}_\theta = -r\sin\theta\vec{E}_1 + r\cos\theta\vec{E}_2 \\ \vec{e}_z = \vec{E}_3 \end{cases}$$

For calculating contravariant basis vectors, we receive $\vec{e}^{\,i} = \dfrac{\partial x^i}{\partial y^j}\vec{E}_j$. After expansion and using the functional relations, we get

$$\begin{cases} \vec{e}^{\,r} = \cos\theta\vec{E}_1 + \sin\theta\,\vec{E}_2 \\ \vec{e}^{\,\theta} = -\dfrac{\sin\theta}{r}\vec{E}_1 + \dfrac{\cos\theta}{r}\vec{E}_2 \\ \vec{e}^{\,z} = \vec{E}_3 \end{cases}$$

The scale factors are the magnitudes of the covariant basis vectors. Hence, $h_1 = h_r = \sqrt{\vec{e}_r \cdot \vec{e}_r} = 1$, $h_2 = h_\theta = \sqrt{\vec{e}_\theta \cdot \vec{e}_\theta} = r$, and $h_3 = h_z = 1$. The unit vectors $\vec{e}(i) = \dfrac{\vec{e}_i}{h_i}$, are

$$\begin{cases} \vec{e}(r) = \cos\theta\vec{E}_1 + \sin\theta\,\vec{E}_2 \\ \vec{e}(\theta) = -\sin\theta\vec{E}_1 + \cos\theta\vec{E}_2 \\ \vec{e}(z) = \vec{E}_3 \end{cases}$$

Note that unit vectors $\vec{e}(r)$ and $\vec{e}(\theta)$ in cylindrical coordinate system change with location and are not constant vectors.

The metric tensor is $g_{ij} = \vec{e}_i \cdot \vec{e}_j$, or $g_{rr} = h_r^2 = 1$, $g_{\theta\theta} = h_\theta^2 = r^2$, $g_{zz} = h_z^2 = 1$, and the off-diagonal elements of the metric tensor are null, hence the coordinate system is orthogonal, or $g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The Jacobian is $\mathcal{J} = h_r h_\theta h_z = r$.

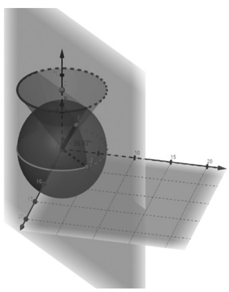
The contravariant metric tensor can be calculated using $g^{ij} = \vec{e}^{\,i} \cdot \vec{e}^{\,j}$, or $g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which is equal to the inverse of covariant metric tensor.

Another way for calculating the contravariant basis vectors is by using the relation $\vec{e}^{\,i} = g^{ij}\vec{e}_j$, or $\vec{e}^{\,1} = g^{11}\vec{e}_1 = \vec{e}_1$, $\vec{e}^{\,2} = g^{22}\vec{e}_2 = r^{-2}\vec{e}_2$, and $\vec{e}^{\,3} = g^{33}\vec{e}_3 = \vec{e}_3$.

For the line element we have, $\left|\overrightarrow{ds}\right| = \sqrt{\left(h_i dx^i\right)^2} = \sqrt{(dr)^2 + (rd\theta)^2 + (dz)^2}$.

8.2 EXAMPLE: SPHERICAL COORDINATE SYSTEMS

Consider a spherical polar coordinate system $\left(x^1, x^2, x^3\right) \equiv (r, \varphi, \theta)$ where r is the radial distance, φ the polar/meridian angle, and θ the azimuthal angle w.r.t. the Cartesian coordinates $\left(y^1, y^2, y^3\right) \equiv (X, Y, Z)$, as shown in Figure 8.3. The functional relations corresponding to spherical and Cartesian systems are:



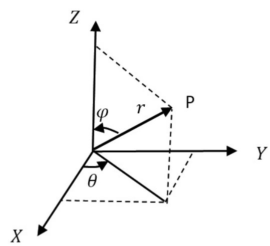


FIGURE 8.3 Spherical coordinate system.

$\begin{cases} X = r\sin\varphi\cos\theta \\ Y = r\sin\varphi\sin\theta \\ Z = r\cos\varphi \end{cases} \equiv \begin{cases} y^1 = x^1\sin x^2\cos x^3 \\ y^2 = x^1\sin x^2\sin x^3 \\ y^3 = x^1\cos x^2 \end{cases}$. Find the inverse relations (i.e., $x^i = F_i\left(y^1, y^2, \cdots, y^N\right)$, the basis vectors $\vec{e}_i$ and $\vec{e}^i$ for the spherical coordinate system in terms of the Cartesian unit vectors. Also find the scale factors and unit vectors and metric tensors g_{ij} and g^{ij} and line element magnitude $\left|\overrightarrow{ds}\right|$.

Solution:

The functions F_i can be obtained by solving for X, Y, Z, using corresponding functional relations, or

$$\begin{cases} r = \sqrt{X^2+Y^2+Z^2} \\ \varphi = \cos^{-1}\left(\dfrac{Z}{\sqrt{X^2+Y^2+Z^2}}\right) \\ \theta = \tan^{-1}\left(\dfrac{Y}{X}\right) \end{cases} \equiv \begin{cases} x^1 = \sqrt{\left(y^1\right)^2+\left(y^2\right)^2+\left(y^3\right)^2} \\ x^2 = \cos^{-1}\left(\dfrac{y^3}{\sqrt{\left(y^1\right)^2+\left(y^2\right)^2+\left(y^3\right)^2}}\right) \\ x^3 = \tan^{-1}\left(\dfrac{y^1}{y^2}\right) \end{cases}.$$

To find the covariant basis vectors, we use $\vec{e}_i = \dfrac{\partial y^j}{\partial x^i}\vec{E}_j$. Therefore, we get $\vec{e}_1 = \dfrac{\partial y^1}{\partial x^1}\vec{E}_1 + \dfrac{\partial y^2}{\partial x^1}\vec{E}_2 + \dfrac{\partial y^3}{\partial x^1}\vec{E}_3 = \sin x^2\cos x^3\vec{E}_1 + \sin x^3\sin x^2\vec{E}_2 + \cos x^2\vec{E}_3$ Similarly, $\vec{e}_2 = \dfrac{\partial y^1}{\partial x^2}\vec{E}_1 + \dfrac{\partial y^2}{\partial x^2}\vec{E}_2 + \dfrac{\partial y^3}{\partial x^2}\vec{E}_3 = x^1\cos x^2\cos x^3\vec{E}_1 + x^1\sin x^3\cos x^2\vec{E}_2 - x^1\sin x^2\vec{E}_3$, and $\vec{e}_3 = \dfrac{\partial y^1}{\partial x^3}\vec{E}_1 + \dfrac{\partial y^2}{\partial x^3}\vec{E}_2 + \dfrac{\partial y^3}{\partial x^3}\vec{E}_3 = -x^1\sin x^2\sin x^3\vec{E}_1 + x^1\cos x^3\sin x^2\vec{E}_2$. In terms of coordinate variables, we have

$$\begin{cases} \vec{e}_r = \sin\varphi\cos\theta\vec{E}_1 + \sin\varphi\sin\theta\vec{E}_2 + \cos\varphi\vec{E}_3 \\ \vec{e}_\varphi = r\cos\varphi\cos\theta\vec{E}_1 + r\cos\varphi\sin\theta\vec{E}_2 - r\sin\varphi\vec{E}_3 \\ \vec{e}_\theta = -r\sin\varphi\sin\theta\vec{E}_1 + r\sin\varphi\cos\theta\vec{E}_2 \end{cases}$$

The scale factors are the magnitudes of the basis vectors. Hence $h_1 = h_r = \sqrt{\vec{e}_r \cdot \vec{e}_r} = 1$, $h_2 = h_\varphi = \sqrt{\vec{e}_\varphi \cdot \vec{e}_\varphi} = r$, and $h_3 = h_\theta = \sqrt{\vec{e}_\theta \cdot \vec{e}_\theta} = r\sin\varphi$. The unit vectors $\vec{e}(i) = \dfrac{\vec{e}_i}{h_i}$, are

$$\begin{cases} \vec{e}(r) = \sin\varphi\cos\theta\vec{E}_1 + \sin\varphi\sin\theta\vec{E}_2 + \cos\varphi\vec{E}_3 \\ \vec{e}(\varphi) = \cos\varphi\cos\theta\vec{E}_1 + \cos\varphi\sin\theta\vec{E}_2 - \sin\varphi\vec{E}_3 \\ \vec{e}(\theta) = -\sin\theta\vec{E}_1 + \cos\theta\vec{E}_2 \end{cases}$$

Note that unit vectors in spherical coordinate system change with location and are not constant vectors. The metric tensor is $g_{ij} = \vec{e}_i \cdot \vec{e}_j$, or $g_{rr} = h_r^2 = 1$, $g_{\varphi\varphi} = h_\varphi^2 = r^2$, $g_{\theta\theta} = h_\theta^2 = r^2\sin^2\varphi$, and the off-diagonal elements of the metric tensor are null, hence the coordinate system is orthogonal, $g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & (r\sin\varphi)^2 \end{bmatrix}$. The Jacobian is $\mathcal{J} = h_r h_\varphi h_\theta = r^2\sin\varphi$. The $g^{ii} = h_i^{-2}$, or $g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & (r\sin\varphi)^{-2} \end{bmatrix}$, then we can calculate the contravariant basis

vectors, as $\vec{e}^{\,j} = g^{ij}\vec{e}_i$, or $\vec{e}^{\,1} = \vec{e}_1$, $\vec{e}^{\,2} = \dfrac{1}{r^2}\vec{e}_2$, and $\vec{e}^{\,3} = \dfrac{1}{r^2\sin^2\varphi}\vec{e}_3$ which yields

$$\begin{cases} \vec{e}^{\,r} = \sin\varphi\cos\theta\vec{E}_1 + \sin\varphi\sin\theta\vec{E}_2 + \cos\varphi\vec{E}_3 \\ \vec{e}^{\,\varphi} = \dfrac{\cos\varphi\cos\theta}{r}\vec{E}_1 + \dfrac{\cos\varphi\sin\theta}{r}\vec{E}_2 - \dfrac{\sin\varphi}{r}\vec{E}_3 \\ \vec{e}^{\,\theta} = -\dfrac{\sin\theta}{r\sin\varphi}\vec{E}_1 + \dfrac{\cos\theta}{r\sin\varphi}\vec{E}_2 \end{cases}$$

For the line element we have, $\left|\overline{ds}\right| = \sqrt{\left(h_i dx^i\right)^2} = \sqrt{(dr)^2 + (rd\varphi)^2 + (r\sin\varphi d\theta)^2}$.

CHAPTER 9

DOT AND CROSS PRODUCTS OF TENSORS

We often must multiply tensors by each other. We consider two vectors for discussion here, without losing generality. When we multiply two scalars (i.e., tensors of rank zero) we just deal with arithmetic multiplication. But a vector has a direction, in addition to its magnitude, that should be taken into consideration for multiplication operations. In principle we can have two vectors just multiply by each other like $\vec{A}\vec{B}$, which is a tensor of rank two, called a *dyadic* product. Or, multiply the vectors' magnitudes and form a new vector with the resulting magnitude directed perpendicular to the plane containing the original vectors, the *cross-product* $\vec{A} \times \vec{B}$, which is a tensor of rank one. Or, multiply the vectors' magnitudes and reduce the rank of the product by projecting one vector on the other one, the *dot-product* $\vec{A} \cdot \vec{B}$, which is a tensor of rank zero. Obviously, for higher than rank-one tensors we would have a greater number of combinations as the result of their products. In general, compared to the rank of original tensors, dyadic product increases the rank, cross-product keeps the rank, and dot-product decreases the rank, i.e., the result is a tensor quantity of higher, the same, or lower rank (by one level/rank) w.r.t to the original quantities, respectively.

We start with dot-product operations. Let's consider two vectors $\vec{A}$ and $\vec{B}$; the dot-product of these two vectors is given by $\vec{A} \cdot \vec{B} = \left(A^i \vec{e}_i\right) \cdot \left(B^j \vec{e}_j\right)$, using the contravariant components of the vectors. We expand this expression, with Equations 7.7 and 7.8, to receive

$$\vec{A} \cdot \vec{B} = A^i B^j g_{ij} = A^i B^j h_i h_j \cos\alpha \qquad 9.1$$

where α is the angle between the two vectors. Further, using Equation 8.1 we receive

$$\vec{A} \cdot \vec{B} = A^i B_i = A_j B^j \qquad 9.2$$

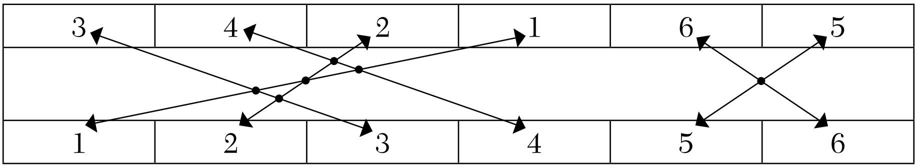
Similarly, we could start with using the covariant components of the vector, but the conclusive results would be the same as the expression given by Equation 9.2. Readers may want to do this as an exercise. From Equation 9.2 we can conclude that the order of multiplication does not change the result of the dot-product, or $A^i B_i = B_i A^i$.

In an orthogonal Cartesian system, either Equation 9.1 or 9.2 would result in $\vec{A} \cdot \vec{B} = A^i B^j \delta_{ij} = A^1 B^1 + A^2 B^2 + \cdots + A^N B^N = A_1 B_1 + A_2 B_2 + \cdots + A_N B_N$. The second equality results from the fact that in orthogonal Cartesian systems there is no distinction between contravariant and covariant components. An example of the dot-product quantity of two vectors is the mechanical work—the dot product of force and distance vectors.

The cross-product operation can be established with a more general formulation using the permutation symbol (also referred to as the Levi-Civita symbol) defined as

$$e_{i_1 i_2 \cdots i_N} = \begin{cases} +1 & \textit{even permutation} \\ -1 & \textit{odd permutation} \\ 0 & \textit{repeated index} \end{cases} \qquad 9.3$$

For example, for $N = 6$ we receive $e_{i_1 i_2 \cdots i_6}$. The permutation is considered even if an even number of interchanges of indices put the indices in arithmetic order (i.e., 123456), and is considered odd if an odd number of interchanges of indices put the indices in order. For any case with a repeated index the permutation symbol is zero. As an example, let's consider e_{342165}. Examining the indices, we find out that 4 interchanges (i.e. $342165 \xrightarrow{(3\leftrightarrow 1)} 142365 \xrightarrow{(4\leftrightarrow 2)} 124365 \xrightarrow{(4\leftrightarrow 3)} 123465 \xrightarrow{(6\leftrightarrow 5)} 123456$) put them in order; note that the sequence of interchanges is irrelevant. Therefore $e_{342165} = 1$. Similarly, $e_{362145} = -1$, since 5 interchanges put the indices in order. If any of the indices is repeated then it is zero, for example $e_{344165} = e_{142165} = 0$, etc. We purposefully called the permutation symbol a *symbol* and not a tensor, for reasons that will be explained later in this section. Another way of identifying the even or odd number of permutations is to write down the given indices as a set and connect them with the equivalent number set but in arithmetic order. Then the number of points at the cross-section of the lines connecting the same numbers is the order of permutation. This method is shown in the following figure for e_{342165}. The connecting lines intersect at six points; hence the permutation is even.



Now considering two vectors $\vec{A}$ and $\vec{B}$ in a Cartesian system the cross-product of these two vectors is another vector $\vec{C}$, given as (we propose this expression, for now at least)

$$C^i = \underbrace{e_{ijk}A^jB^k}_{A\times B} = -\underbrace{e_{ijk}B^jA^k}_{B\times A}, \mathit{Cartesian} \tag{9.4}$$

for the i-component of $\vec{C}$. This relation clearly shows that the order of multiplication matters for cross-product operation. For example, in a 3D Cartesian system we have $\vec{A} = \left(A^1, A^2, A^3\right)$ and $\vec{B} = \left(B^1, B^2, B^3\right)$ and we find their cross-product using Equation 9.4, $\vec{C} = \left(C^1, C^2, C^3\right) = \vec{A}\times\vec{B}$ as, $C^1 = A^2B^3 - A^3B^2$, $C^2 = A^3B^1 - A^1B^3$, and $C^3 = A^1B^2 - A^2B^1$.

Note that in Equation 9.4, we have written the relation for the contravariant component of the resulting vector $\vec{C}$. As we mentioned previously, in a Cartesian system contravariant or covariant components need not be considered; hence we usually place all indices as subscripts. But in an arbitrary system we should make sure that the resulting C^i is transformed like a contravariant component/quantity. To show this, we set guidelines to form a tensor expression in an arbitrary coordinate system, x^i as follows:

a. Identify the rank of the expression representing the desired quantity in terms of a tensor; i.e., zero is a scalar, one is a vector, two or more a tensor of the corresponding rank.

b. Identify and decide if the expression should be in contravariant or covariant form and place the indices appropriately (i.e., as subscript for covariant and superscript for contravariant).

c. Form the expression based on guidelines (a) and (b).

d. Transform the expression from arbitrary system x^i to another system, say x'^i (or vice versa) using the linear operators $\dfrac{\partial x'^i}{\partial x^j}$ or $\dfrac{\partial x^i}{\partial x'^j}$ according to the combination rule (see Sections 4 and 5).

e. Show that the resulting expression recovers the related familiar form in Cartesian coordinates.

Usually, starting from an expression form in a Cartesian system (as listed in part (e), above) gives a reasonable and usually the correct form to start with. Let's apply these guidelines to the expression given by Equation 9.4. By observing the expression, we conclude that the contravariant components A^j and B^k are correct forms since they transform like contravariant quantities. However, for the permutation symbol we need to make sure that it is a tensor quantity, i.e., it transforms like a tensor. It can be shown that the permutation symbol does not transform like a tensor unless a necessary adjustment is implemented [2], [4], [7] by multiplying it by the Jacobian of the transformation (see Equation 4.4). Now, we define the covariant and contravariant *permutation tensors* by multiplying Jacobian and inverse of Jacobian to the permutation symbol, respectively, given as

$$\begin{cases} \mathcal{E}_{i_1 i_2 \cdots i_N} = \mathcal{J} e_{i_1 i_2 \cdots i_N} \\ \mathcal{E}^{i_1 i_2 \cdots i_N} = \mathcal{J}^{-1} e_{i_1 i_2 \cdots i_N} \end{cases} \tag{9.5}$$

Therefore, the general expression for the vector $\vec{C}$, for example, is $C_i = \mathcal{J} e_{ijk} A^j B^k$ and $C^i = \mathcal{J}^{-1} e_{ijk} A_j B_k$. Using permutation tensor, we will get

$$\begin{cases} C_i = \mathcal{E}_{ijk} A^j B^k = \mathcal{J} e_{ijk} A^j B^k \quad \textit{covariant component} \\ C^i = \mathcal{E}^{ijk} A_j B_k = \mathcal{J}^{-1} e_{ijk} A_j B_k \; \textit{contravariant component} \end{cases} \tag{9.6}$$

Note that in Equation 9.6, for the covariant (contravariant) component of the resulting vector $\vec{C}$, we used the combination of covariant (contravariant) permutation tensor and contravariant (covariant) components of the two contributing vectors $\vec{A}$ and $\vec{B}$.

To show that, for example, $\mathcal{E}_{ijk} A^j B^k$ transforms like a covariant quantity, we write down its transformation between two arbitrary systems, or $\mathcal{E}'_{ijk} A'^j B'^k = \left(\mathcal{E}_{lmn} \dfrac{\partial x^l}{\partial x'^i} \dfrac{\partial x^m}{\partial x'^j} \dfrac{\partial x^n}{\partial x'^k} \right) \left(A^p \dfrac{\partial x'^j}{\partial x^p} \right) \left(B^q \dfrac{\partial x'^k}{\partial x^q} \right) = \mathcal{E}_{lmn} A^p B^q \dfrac{\partial x^l}{\partial x'^i} \delta^m_p \delta^n_q =$ $\left(\mathcal{E}_{lmn} A^m B^n \right) \dfrac{\partial x^l}{\partial x'^i}$. Similarly, for the contravariant form (i.e., the second expression in Equation 9.6) it can be shown that it does indeed transform like a contravariant quantity. Readers may want to do this as an exercise.

The cross-product of two vectors can be written in terms of their physical components as well. This can be done using, for example, the contravariant component when multiplied by the corresponding scale factor. Therefore, we can write for an orthogonal system (using Equations 9.5, 9.6, and 11.8)

$$\underbrace{h_i C^i}_{C(i)} = h_i \mathcal{E}^{ijk} A_j B_k \text{, or } C(i) = \frac{h_i e_{ijk}}{\mathcal{J}} A_j B_k = \frac{h_i e_{ijk}}{\mathcal{J}} \left[h_j h_k A(j) B(k) \right] \text{, or}$$

$$C(i) = \frac{e_{ijk}}{\mathcal{J}} h_i h_j h_k \left[A(j) B(k) \right] ,\mathit{orthogonal} \qquad 9.7$$

For orthogonal coordinate systems, using $\mathcal{J} = h_1 h_2 h_3$, we receive

$$\begin{cases} \left(\vec{A} \times \vec{B}\right)(1) = C(1) = \left[A(2)B(3) - A(3)B(2)\right] \\ \left(\vec{A} \times \vec{B}\right)(2) = C(2) = \left[A(3)B(1) - A(1)B(3)\right] \\ \left(\vec{A} \times \vec{B}\right)(3) = C(3) = \left[A(1)B(2) - A(2)B(1)\right] \end{cases} \qquad 9.8$$

Expanding on these results, in a N-dimensional space, the generalized cross-product of $N-1$ vectors reads, using Equation 9.6,

$$\begin{cases} C_{i_1} = \mathcal{E}_{i_1 i_2 \cdots i_N} A^{i_2} B^{i_3} \cdots Z^{i_N} \\ C^{i_1} = \mathcal{E}^{i_1 i_2 \cdots i_N} A^{i_2} B^{i_3} \cdots Z^{i_N} \end{cases} \qquad 9.9$$

where now vector $\vec{C}$ is perpendicular to the hyperplane formed by the $N-1$ vectors $\vec{A}$ through $\vec{Z}$.

Immediately, using Equation 9.9 we can conclude that in an arbitrary system the unit volume $d\mathbb{V}$ is, [2],

$$d\mathbb{V} = \mathcal{J} dx^1 dx^2 \cdots dx^N \qquad 9.10$$

For example, in the spherical system (r, φ, θ) we have $d\mathbb{V} = r^2 \sin\varphi \, dr d\varphi d\theta$. Another useful extension is for the orthogonal systems (i.e., $g_{i \neq j} = 0$), for which we get $\left|g_{ij}\right| = \prod g_{ii} = g_{11} g_{22} \cdots g_{NN}$. But $g_{11} = \vec{e}_1 \cdot \vec{e}_1 = h_1^2$, $g_{22} = \vec{e}_2 \cdot \vec{e}_2 = h_2^2$ and so forth. Hence, $\left|g_{ij}\right| = h_1^2 h_2^2 \cdots h_N^2$. Using the relationship $\left|g_{ij}\right| = \mathcal{J}^2$, we finally reach at

$$\mathcal{J} = \prod_i h_i = h_1 h_2 \cdots h_N \;\; \mathit{orthogonal\ system} \qquad 9.11$$

And hence

$$d\mathbb{V} = \prod_i h_i dx^i = \left(h_1 dx^1\right)\left(h_2 dx^2\right) \cdots \left(h_N dx^N\right) \;\; \mathit{orthogonal\ system} \qquad 9.12$$

9.1 DETERMINANT OF AN $N \times N$ MATRIX USING PERMUTATION SYMBOLS

Using the permutation symbol e_{ijk} (see Equation 9.3), we can write the determinant of a 3×3 matrix $[M]$, as $|M| = e_{ijk} M_{1i} M_{2j} M_{3k} = e_{ijk} M_{i1} M_{j2} M_{k3}$, given by expansion based on row or column, respectively, and M_{ij} represent the elements of the matrix [2], [4]. To expand on these results, we consider $[M]$ to be an $N \times N$ matrix, $(N \geq 3)$. Therefore, we can write its determinant as

$$|M| = e_{i_1 i_2 \cdots i_N} M_{1 i_1} M_{2 i_2} \cdots M_{N i_N} = e_{i_1 i_2 \cdots i_N} M_{i_1 1} M_{i_2 2} \cdots M_{i_N N} \qquad 9.13$$

See the application example in Section 16.10.

CHAPTER 10

GRADIENT VECTOR OPERATOR—CHRISTOFFEL SYMBOLS

In engineering and science, most if not all governing equations (e.g., equilibrium, Navier-Stokes, Maxwell's equations) contain terms that involve derivatives of tensor quantities. These equations are mathematical models of related quantity transports, like momentum, mass, energy, etc. Different forms of derivatives result from the gradient operator, which is a vector that takes the derivative of and performs dyadic, dot-product, cross-product, etc. operations on the involved tensors. Depending on the rank of the tensors under the gradient operation, the result could be complex expressions. In the previous sections, we have defined and derived necessary formulae to tackle the derivation of gradient vector transformation and consequently find the general forms of related derivatives appearing in the governing equations.

10.1 COVARIANT DERIVATIVES OF VECTORS—CHRISTOFFEL SYMBOLS OF THE 2ND KIND

The covariant component of the gradient vector $\vec{\nabla}$ is defined as $\nabla_i = \dfrac{\partial}{\partial x^i}$, or we can write the full vector as $\vec{\nabla} = \vec{e}^{\,i} \dfrac{\partial}{\partial x^i} = \vec{e}^{\,i} \nabla_i$. To show that ∇_i transforms like a covariant component between two arbitrary systems, we can write $\nabla'_i = \dfrac{\partial}{\partial x'^i} = \dfrac{\partial x^j}{\partial x'^i} \dfrac{\partial}{\partial x^j} = \dfrac{\partial x^j}{\partial x'^i} \nabla_j$. This expression clearly shows that ∇'_i is a covariant quantity. The transformation from x'^i to x^i is obvious.

When a gradient operates on a scalar, or tensor, of rank zero like temperature, pressure, or concentration, we can write it as $\nabla_i \psi$ and the corresponding transformation is $\nabla'_i \psi' = \nabla'_i \psi = \frac{\partial x^j}{\partial x'^i} \nabla_j \psi$, using the fact that scalar ψ is invariant (or $\psi = \psi'$) and independent of the coordinate systems selected. Now, assuming that the gradient operates on a vector, say $\vec{A}$, which is also an invariant quantity, the result is $\vec{\nabla}\vec{A}$, or a tensor of rank two. The covariant component is then $\nabla_i \vec{A}$, which is a vector by itself. Expanding, we get

$$\nabla_i \vec{A} = \nabla_i \left(A^j \vec{e}_j \right) = \vec{e}_j \left(\nabla_i A^j \right) + A^j \left(\nabla_i \vec{e}_j \right) \qquad 10.1$$

The first term in the R.H.S of Equation 10.1 is the straight-forward derivative of the component A^j. But the second term involves a gradient of the basis vector $\vec{e}_j$, which is not necessarily zero in an arbitrary system since the basis vector's magnitude and/or direction may change from point to point. Note that in a Cartesian system, $\nabla_i \vec{E}_j = 0$, i.e., the second term in Equation 10.1, vanishes. Therefore, we can interpret $\nabla_i \vec{e}_j$ as a measure of the curvature of the arbitrary coordinate surface $x^i = constant$ (e.g., in a spherical coordinate system $X^2 + Y^2 + Z^2 = r^2$ is the surface of the sphere). The physical meaning of $\nabla_i \vec{e}_j$ is this that it represents the rate of change of $\vec{e}_j$ with respect to coordinate variable x^i in the direction of basis vector $\vec{e}_k$. This quantity is represented by Γ^k_{ij} (some authors use $\begin{Bmatrix} k \\ ij \end{Bmatrix}$ instead) and is the *Christoffel symbol of the second kind* [8].

It may be useful to pause at this point and make sure that the physical/geometrical meaning of the Christoffel symbol is well understood. To help with this understanding, we consider a simple 2D polar coordinate system, $(r, \theta) \equiv (x^1, x^2)$ as shown in Figure 10.1.

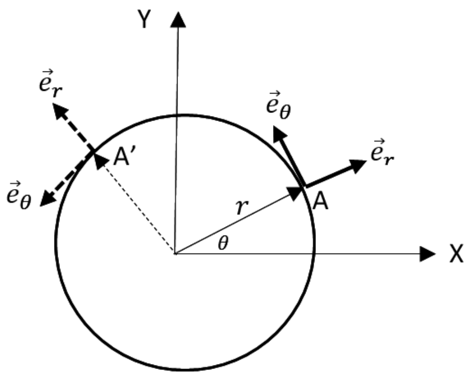


FIGURE 10.1 2D Polar coordinate system.

As shown, the covariant basis vectors $\vec{e}_r$ and $\vec{e}_\theta$ change from point to point in space, e.g., points A and A', in general. Let's consider, for example $\nabla_\theta \vec{e}_\theta = \frac{\partial \vec{e}_\theta}{\partial \theta}$ which represents the change for basis vector $\vec{e}_\theta$ with respect to coordinate variable θ. This derivative is a vector and has two components in the (r,θ) coordinate system. Therefore, we can write it as a linear combination of the basis vectors in polar coordinate system $\vec{e}_r$ and $\vec{e}_\theta$, or

$$\nabla_\theta \vec{e}_\theta = \frac{\partial \vec{e}_\theta}{\partial \theta} = \alpha \vec{e}_r + \beta \vec{e}_\theta \tag{10.2}$$

Similar expressions can be written down for other possible derivatives (i.e., $\nabla_\theta \vec{e}_r$, $\nabla_r \vec{e}_\theta$, $\nabla_r \vec{e}_r$). In our example (see Equation 10.2), the coefficients α and β are the corresponding Christoffel symbols, also referred to as *connection coefficients*. α is the component of the vector $\frac{\partial \vec{e}_\theta}{\partial \theta}$ in the direction of $\vec{e}_r$, or $\Gamma^r_{\theta\theta}$ and β is the component of the vector $\frac{\partial \vec{e}_\theta}{\partial \theta}$ in the direction of $\vec{e}_\theta$, or $\Gamma^\theta_{\theta\theta}$. Using these definitions, we can write all possible derivatives with their corresponding Christoffel symbols as

$$\begin{cases} \dfrac{\partial \vec{e}_\theta}{\partial \theta} = \Gamma^r_{\theta\theta}\vec{e}_r + \Gamma^\theta_{\theta\theta}\vec{e}_\theta \\ \dfrac{\partial \vec{e}_r}{\partial \theta} = \Gamma^r_{r\theta}\vec{e}_r + \Gamma^\theta_{r\theta}\vec{e}_\theta \\ \dfrac{\partial \vec{e}_\theta}{\partial r} = \Gamma^r_{\theta r}\vec{e}_r + \Gamma^\theta_{\theta r}\vec{e}_\theta \\ \dfrac{\partial \vec{e}_r}{\partial r} = \Gamma^r_{rr}\vec{e}_r + \Gamma^\theta_{rr}\vec{e}_\theta \end{cases} \tag{10.3}$$

The values of Christoffel symbols can be calculated using the functional relations between the polar and Cartesian systems. For this purpose, we need the relations for the basis vectors. Using Equation 5.3, we have $\vec{e}_r = \frac{\partial X}{\partial r}\vec{E}_1 + \frac{\partial Y}{\partial r}\vec{E}_2$ and $\vec{e}_\theta = \frac{\partial X}{\partial \theta}\vec{E}_1 + \frac{\partial Y}{\partial \theta}\vec{E}_2$, where $\vec{E}_i$ are unit vectors in Cartesian system. Performing the calculations, using the functional relation between the polar and Cartesian systems, as $X = r\cos\theta$ and $Y = r\sin\theta$, we receive

$$\begin{cases} \vec{e}_r = \cos\theta \vec{E}_1 + \sin\theta \vec{E}_2 \\ \vec{e}_\theta = -r\sin\theta \vec{E}_1 + r\cos\theta \vec{E}_2 \end{cases} \tag{10.4}$$

Using Equations 10.4 we receive $\frac{\partial \vec{e}_\theta}{\partial \theta} = -r\cos\theta \vec{E}_1 - r\sin\theta \vec{E}_2 = -r\vec{e}_r$, $\frac{\partial \vec{e}_r}{\partial \theta} = -\sin\theta \vec{E}_1 + \cos\theta \vec{E}_2 = \frac{1}{r}\vec{e}_\theta$, $\frac{\partial \vec{e}_\theta}{\partial r} = -\sin\theta \vec{E}_1 + \cos\theta \vec{E}_2 = \frac{1}{r}\vec{e}_\theta$, and $\frac{\partial \vec{e}_r}{\partial r} = 0$. Finally, comparing these results with Equation 10.3, we find the Christoffel symbols values for polar coordinate system as $\Gamma^r_{\theta\theta} = -r$, $\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}$, and $\Gamma^\theta_{\theta\theta} = \Gamma^\theta_{rr} = \Gamma^r_{\theta r} = \Gamma^r_{r\theta} = \Gamma^r_{rr} = 0$. Table 10.1 gives the summary of these results and the matrix form of these symbols for polar coordinate systems.

TABLE 10.1 Christoffel symbols of the 2nd kind for a 2D polar coordinate system (r,θ).

Symbol	Meaning	Value	Symmetry	Matrix form
$\Gamma^r_{\theta\theta}$	$\vec{e}_\theta$ change w.r.t θ in $\vec{e}_r$ direction	$-r$	–	$\Gamma^r = \begin{bmatrix} \Gamma^r_{rr} & \Gamma^r_{r\theta} \\ \Gamma^r_{\theta r} & \Gamma^r_{\theta\theta} \end{bmatrix}$
$\Gamma^r_{\theta r}$	$\vec{e}_\theta$ change w.r.t r in $\vec{e}_r$ direction	0	$=\Gamma^r_{r\theta}$	
Γ^r_{rr}	$\vec{e}_r$ change w.r.t r in $\vec{e}_r$ direction	0	–	
$\Gamma^\theta_{\theta\theta}$	$\vec{e}_\theta$ change w.r.t θ in $\vec{e}_\theta$ direction	0	–	$\Gamma^\theta = \begin{bmatrix} \Gamma^\theta_{rr} & \Gamma^\theta_{r\theta} \\ \Gamma^\theta_{\theta r} & \Gamma^\theta_{\theta\theta} \end{bmatrix}$
$\Gamma^\theta_{r\theta}$	$\vec{e}_r$ change w.r.t θ in $\vec{e}_\theta$ direction	$1/r$	$=\Gamma^\theta_{\theta r}$	
Γ^θ_{rr}	$\vec{e}_r$ change w.r.t r in $\vec{e}_\theta$ direction	0	–	

The number of independent Christoffel symbols can be obtained using $\frac{N^2(N+1)}{2}$, for an N-dimensional system. For example, in a $2D$ polar coordinate system (r,θ) with $N=2$, we receive $\frac{4\times 3}{2} = 6$ symbols, among them only two are non-zero. Note that Christoffel symbols are symmetric with respect to the lower indices (see the next section).

Now we continue with our general formulation and discussion, after establishing the physical/geometrical meaning of Christoffel symbols of 2nd kind with this example.

Rewriting $\nabla_i \vec{e}_j$ using the definition of Christoffel symbol (see Equation 10.3), we have

$$\nabla_i \vec{e}_j = \Gamma^k_{ij} \vec{e}_k \qquad 10.5$$

Plugging back into Equation 10.1, we receive $\nabla_i \vec{A} = \vec{e}_j \left(\nabla_i A^j\right) + A^j \Gamma^k_{ij} \vec{e}_k$. In this relation, for the last term we interchange j and k indices, since they are dummy indices. Then, $\nabla_i \vec{A} = \vec{e}_j \left(\nabla_i A^j\right) + A^k \Gamma^j_{ik} \vec{e}_j = \vec{e}_j \left(\nabla_i A^j + \Gamma^j_{ik} A^k\right)$. The expression in the bracket is a tensor, since both $\nabla_i \vec{A}$ and $\vec{e}_j$ are tensors. We define the expression in the bracket as $A^j_{,i}$ where the comma in the subscript indicates differentiation and index i represents the covariant component of gradient vector. Hence,

$$A^j_{,i} = \frac{\partial A^j}{\partial x^i} + \Gamma^j_{ik} A^k \qquad 10.6$$

Equation 10.6 is the covariant derivative of the contravariant component of vector $\vec{A}$. Or

$$\nabla_i \vec{A} = A^j_{,i} \vec{e}_j \qquad 10.7$$

10.2 CONTRAVARIANT DERIVATIVES OF VECTORS

At this point in our discussion, it seems logical to seek the contravariant component of gradient vector, ∇^i. We can write this operator as $\nabla^i = g^{ij} \nabla_j$ using a metric tensor (see Section 8). Therefore, we can raise the derivative index by multiplying the contravariant metric tensor to the expression, or

$$A^{k\,i}_{,\,\cdot} = g^{ij} A^k_{,j} \qquad 10.8$$

Equation 10.8 is the contravariant derivative of the contravariant component of vector $\vec{A}$. Or

$$\nabla^i \vec{A} = A^{k\,i}_{,\,\cdot} \vec{e}_k \qquad 10.9$$

From the start we could use the covariant component of vector $\vec{A}$, to find the corresponding covariant and contravariant gradient components as well (see Equation 10.1). To this end, we can write the covariant derivative of the covariant component of vector $\vec{A}$, as

$$\nabla_i \vec{A} = \nabla_i \left(A_j \vec{e}^{\,j}\right) = \vec{e}^{\,j} \left(\nabla_i A_j\right) + A_j \left(\nabla_i \vec{e}^{\,j}\right) \qquad 10.10$$

The first term on the R.H.S of Equation 10.10 is straight-forward, the derivative of the component A_j. The second term involves the gradient of the contravariant basis vector $\vec{e}^{\,j}$, for which we require a new expression. To find this quantity, we use Equation 10.6 to write $\vec{e}^{\,j}_{,i} = \frac{\partial \vec{e}^{\,j}}{\partial x^i} + \Gamma^j_{ik}\vec{e}^{\,k}$. Due to the fact that $\vec{e}^{\,j}_{,i}$ is a tensor (this can be shown by transformation of the terms involved between arbitrary systems, like $A^j_{,i}$) and its value in a Cartesian system is null, hence it is equal to zero in any arbitrary coordinate system. Therefore, $\vec{e}^{\,j}_{,i} = \frac{\partial \vec{e}^{\,j}}{\partial x^i} + \Gamma^j_{ik}\vec{e}^{\,k} = 0$, or

$$\nabla_i \vec{e}^{\,j} = \frac{\partial \vec{e}^{\,j}}{\partial x^i} = -\Gamma^j_{ik}\vec{e}^{\,k} \qquad 10.11$$

Substituting for $\nabla_i \vec{e}^{\,j}$ from Equation 10.11, into Equation 10.10, we receive $\nabla_i\left(A_j\vec{e}^{\,j}\right) = \vec{e}^{\,j}\left(\nabla_i A_j\right) - A_j\left(\Gamma^j_{ik}\vec{e}^{\,k}\right)$. In this relation for the last term we interchange j and k indices, since they are dummy indices. Then, after factoring out $\vec{e}^{\,j}$, we get $\nabla_i\left(A_j\vec{e}^{\,j}\right) = \vec{e}^{\,j}\left(\nabla_i A_j - A_k\Gamma^k_{ij}\right)$. We define the expression in the bracket as $A_{j,i}$ where the comma in subscript indicates differentiation and i represents the covariant component of gradient vector. Hence

$$A_{j,i} = \frac{\partial A_j}{\partial x^i} - \Gamma^k_{ij}A_k \qquad 10.12$$

Equation 10.12 is the covariant derivative of the covariant component of vector $\vec{A}$. Or

$$\nabla_i \vec{A} = A_{j,i}\vec{e}^{\,j} \qquad 10.13$$

10.3 COVARIANT DERIVATIVES OF A MIXED TENSOR

Comparing Equations 10.6 and 10.12, we observe that for each index a term involving a Christoffel symbol is added to the expression on the R.H.S and when a covariant component of a vector is involved in a derivative operation, a minus sign is multiplied to the corresponding Christoffel symbol, contrary to a plus sign for when the contravariant component is involved. This rule can be used to extend the derivative calculation of a tensor of higher ranks. To show this, we consider a mixed tensor of third rank, for example A^{ij}_k with the invariant $A = A^{ij}_k\vec{e}_i\vec{e}_j\vec{e}^{\,k}$. The covariant derivative can be written as

$$A^{ij}_{k,n} = \frac{\partial A^{ij}_k}{\partial x^n} + \Gamma^i_{nm}A^{jm}_k + \Gamma^j_{nm}A^{im}_k - \Gamma^m_{nk}A^{ij}_m \qquad 10.14$$

Therefore, $\nabla_n A = A^{ij}_{k,n}\vec{e}_i\vec{e}_j\vec{e}^{\,k}$ which is the component n of a tensor of rank four (i.e., $\vec{\nabla} A = A^{ij}_{k,n}\vec{e}_i\vec{e}_j\vec{e}^{\,k}\vec{e}^{\,n}$).

10.4 CHRISTOFFEL SYMBOL RELATIONS AND PROPERTIES—1ST AND 2ND KINDS

The definition of a Christoffel symbol of the first kind, properties, and some relations for calculation of Christoffel symbols are discussed in this section. The properties are:

1. Symmetry of the lower indices: The Christoffel symbol of the second kind is symmetric w.r.t lower indices, or

$$\Gamma_{ij}^{k} = \Gamma_{ji}^{k} \qquad 10.15$$

To show this, we use Equation 10.5, after changing index $k \to n$, and perform a dot product operation on it by contravariant basis vector $\vec{e}^{k}$. Or $\vec{e}^{k} \cdot \nabla_i \vec{e}_j = \Gamma_{ij}^{n} \underbrace{\vec{e}^{k} \cdot \vec{e}_n}_{\delta_n^k} = \Gamma_{ij}^{k}$. Therefore, we can write

$$\Gamma_{ij}^{k} = \vec{e}^{k} \cdot \left(\nabla_i \vec{e}_j\right) \qquad 10.16$$

We can use Equation 10.16 to calculate Christoffel symbol Γ_{ij}^{k} in Cartesian coordinate system variables y^i. After expanding the R.H.S expression by appropriate transformation and using chain rule, we will receive $\vec{e}^{k} \cdot \left(\nabla_i \vec{e}_j\right) = \left(\frac{\partial x^k}{\partial y^m}\vec{E}^m\right) \cdot \frac{\partial y^n}{\partial x^i}\frac{\partial}{\partial y^n}\left(\frac{\partial y^p}{\partial x^j}\vec{E}_p\right) = \frac{\partial x^k}{\partial y^m}\frac{\partial y^n}{\partial x^i}\frac{\partial}{\partial y^n}\frac{\partial y^p}{\partial x^j}\underbrace{(\vec{E}^m \cdot \vec{E}_p)}_{\delta_p^m} =$ $\frac{\partial x^k}{\partial y^p}\underbrace{\left(\frac{\partial y^n}{\partial x^i}\frac{\partial x^q}{\partial y^n}\right)}_{\delta_i^q}\frac{\partial^2 y^p}{\partial x^q \partial x^j}$. Where, in the last expression we inserted ($\frac{\partial x^q}{\partial x^q} = 1$, or using chain rule). Finally, we receive

$$\Gamma_{ij}^{k} = \frac{\partial x^k}{\partial y^p}\frac{\partial^2 y^p}{\partial x^i \partial x^j} \qquad 10.17$$

Equation 10.17 clearly shows the symmetry of Γ_{ij}^{k} for its lower indices, since the order of differentiation is irrelevant, i.e., $\frac{\partial^2 y^p}{\partial x^i \partial x^j} = \frac{\partial^2 y^p}{\partial x^j \partial x^i}$.

2. Christoffel symbol is not a tensor: Because its value in a Cartesian system is zero; hence, if it transforms as a tensor it should be zero in any arbitrary system as well. But we know that this is not the case (see Equations 10.10 and 10.17).

3. Covariant derivative of the metric tensor: This quantity leads to a useful formula for calculating the value of a Christoffel symbol. First, we find the covariant derivative of the metric tensor, or $\nabla_i g_{jk} = \nabla_i\left(\vec{e}_j \cdot \vec{e}_k\right) = \vec{e}_j \cdot \left(\nabla_i \vec{e}_k\right) + \left(\nabla_i \vec{e}_j\right) \cdot \vec{e}_k$. Now, using Equation 10.5, we can write $\nabla_i \vec{e}_k = \Gamma^n_{ik}\vec{e}_n$ and $\nabla_i \vec{e}_j = \Gamma^n_{ij}\vec{e}_n$. Therefore, $\nabla_i g_{jk} = \Gamma^n_{ik}\underbrace{(\vec{e}_j \cdot \vec{e}_n)}_{g_{jn}} + \Gamma^n_{ij}\underbrace{(\vec{e}_n \cdot \vec{e}_k)}_{g_{nk}}$ and finally we receive

$$\nabla_i g_{jk} = \frac{\partial g_{jk}}{\partial x^i} = \Gamma^n_{ik}\, g_{jn} + \Gamma^n_{ij}\, g_{nk} \tag{10.18}$$

Note that by comparing Equation 10.18 with the rule of covariant derivatives of tensors (see Equation 10.14) we can conclude that $g_{jk,i} = 0$, a reasonable result considering that a metric tensor in a Cartesian system is a constant/unity and hence its derivative is equal to zero in an arbitrary system.

4. Christoffel symbol of the first kind: So far, we have defined the Christoffel symbol of the second kind. We now derive formulas that can be used for calculating its value as well as defining a Christoffel symbol of the first kind. To do this, we manipulate Equation 10.18 by permuting the indices (i.e., $i \to j, j \to k, \text{and}\, k \to i$), twice in sequence. Hence we get two alternative but equivalent relations as $\frac{\partial g_{ki}}{\partial x^j} = \Gamma^n_{ji}\, g_{jn} + \Gamma^n_{jk}\, g_{ni}$ and $\frac{\partial g_{ij}}{\partial x^k} = \Gamma^n_{kj}\, g_{in} + \Gamma^n_{ki}\, g_{nj}$. Now we subtract Equation 10.18 from the sum of the latter two relations, using along the way the symmetry of both metric tensor and Christoffel symbols. The result reads

$$\Gamma^n_{ij}\, g_{nk} = \frac{1}{2}\left[\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right] \tag{10.19}$$

In Equation 10.19, the quantity $\Gamma^n_{ij}\, g_{nk}$ is the *Christoffel symbol of the first kind*, also written as $\Gamma_{ij,k} \equiv [ij,k]$. We can use $\Gamma^n_{ij}\, g_{nk}$ to find a relation for calculation of the Christoffel symbol of the second kind in terms of metric tensor. To do this we multiply it by g^{mk} to get $\Gamma^n_{ij}\, g_{nk} g^{mk} = \delta^m_n \Gamma^n_{ij} = \Gamma^m_{ij}$. Therefore, after multiplying Equation 10.19 by g^{mk}, we get

$$\Gamma^m_{ij} = \frac{g^{mk}}{2}\left[\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right] \tag{10.20}$$

Equation 10.20 states that the Christoffel symbol of the second kind is equal to the first kind multiplied by the doubly contravariant metric tensor related to the coordinate system in use. Observing Equation 10.19, a symmetry shows up. That is, the derivative is w.r.t. the coordinates indices and for each term in the bracket the remaining indices are used for the metric tensor.

5. For an orthogonal coordinate system we can simplify Equation 10.20, using the property of g_{ij} and g^{ij} being diagonal tensors or matrices (see Equation 7.9). In other words, we can write $g_{ij} = g^{ij} = 0$ for $i \neq j$. Therefore, examining the terms on the R.H.S of Equation 10.20, we can conclude that $\frac{\partial g_{ij}}{\partial x^k} = 0$ and $\frac{\partial g_{ik}}{\partial x^j} \neq 0$ only when $i = k$. Also $g^{mk} \neq 0$ when $m = k$ and $\frac{\partial g_{jk}}{\partial x^i} \neq 0$, since j cannot be equal to k. By implementing these relations among the indices (i.e., $i = k = m$ and $i \neq j$) and rewriting Equation 10.20, we receive

$$\Gamma^i_{ij} = \frac{g^{ii}}{2}\frac{\partial g_{ii}}{\partial x^j}, \qquad \textit{no summmation on i} \qquad 10.21$$

Equation 10.20, for $i = j \neq k = m$ reduces to

$$\Gamma^k_{ii} = -\frac{g^{kk}}{2}\frac{\partial g_{ii}}{\partial x^k}, \qquad \textit{no summmation on i and k} \qquad 10.22$$

For orthogonal coordinate systems, using Equation 7.13, we substitute for $g_{ii} = h_i^2$ and $g^{ii} = h_i^{-2}$ in Equation 10.21 to obtain $\Gamma^i_{ij} = \frac{h_i^{-2}}{2}\frac{\partial\left(h_i^2\right)}{\partial x^j}$, or

$$\Gamma^i_{ij} = \frac{1}{h_i}\frac{\partial h_i}{\partial x^j}, \qquad \textit{no summmation on i, (orthogonal)} \qquad 10.23$$

Simply, by letting $i = j$ in Equation 10.23, we receive

$$\Gamma^i_{ii} = \frac{1}{h_i}\frac{\partial h_i}{\partial x^i}, \qquad \textit{no summmation on i, (orthogonal)} \qquad 10.24$$

Similarly, Equation 10.22 yields $\Gamma^k_{ii} = -\frac{h_k^{-2}}{2}\frac{\partial\left(h_i^2\right)}{\partial x^k}$, or

$$\Gamma^k_{ii} = -\frac{h_i}{h_k^2}\frac{\partial h_i}{\partial x^k}, \qquad \textit{no summmation on i and k, (orthogonal)} \qquad 10.25$$

6. Another useful relation can be derived for the value of Γ^i_{in}, which is $\vec{e}_i$ change w.r.t x^n in the direction of $\vec{e}_i$. We use the fact that the derivative of the permutation tensor, or $\varepsilon^{ijk}_{,n}$ is zero, since its value in the Cartesian coordinates is null. Using Equation 10.14, we can write $\mathcal{E}^{ijk}_{,n} = \dfrac{\partial \mathcal{E}^{ijk}}{\partial x^n} + \Gamma^i_{nm}\mathcal{E}^{mjk} + \Gamma^j_{nm}\mathcal{E}^{imk} + \Gamma^k_{nm}\mathcal{E}^{ijm} = 0$. To simplify the derivation, without losing generality, we select $i=1$, $j=2$, $k=3$. Hence, we get $\dfrac{\partial \mathcal{E}^{123}}{\partial x^n} = -\left(\Gamma^1_{nm}\mathcal{E}^{m23} + \Gamma^2_{nm}\mathcal{E}^{1m3} + \Gamma^3_{nm}\mathcal{E}^{12m}\right)$. Collecting non-zero terms, we have $\dfrac{\partial \mathcal{E}^{123}}{\partial x^n} = -\left(\Gamma^1_{n1}\mathcal{E}^{123} + \Gamma^2_{n2}\mathcal{E}^{123} + \Gamma^3_{n3}\mathcal{E}^{123}\right) = -\mathcal{E}^{123}\Gamma^i_{ni}$, summation applies on i. Recalling that $\mathcal{E}^{123} = 1/\mathcal{J}$ and $\Gamma^i_{ni} = \Gamma^i_{in}$, we get $\dfrac{\partial}{\partial x^n}\left(\dfrac{1}{\mathcal{J}}\right) = -\dfrac{1}{\mathcal{J}}\Gamma^i_{in}$ (recall $\mathcal{J}$ is the Jacobian). Performing the differentiation and rearranging terms, we receive

$$\Gamma^i_{in} = \frac{1}{\mathcal{J}}\frac{\partial \mathcal{J}}{\partial x^n} \qquad 10.26$$

Note that Equation 10.26 works for arbitrary coordinate systems and it recovers Equation 10.23 when the system is orthogonal.

10.4.1. Example: Christoffel symbols for cylindrical and spherical coordinate systems

In this example, we calculate and derive the relations for Christoffel symbols of the second kind, for cylindrical and spherical coordinate systems. The results are also expressed in terms of unit vectors (i.e., the physical components) for each coordinate system. Readers should note that the results depend on the order of coordinate axes, defined. In other words, attention should be given when reading comparable results from other references in relation to their defined corresponding coordinates axes.

Cylindrical coordinates

From the results of Example 8.1, for a cylindrical coordinate system (r,θ,z) (note the order of the coordinates defined here, see Figure 8.2) we can write the non-zero terms of the metric tensor, as $g_{rr} = h_r^2 = 1$, $g_{\theta\theta} = h_\theta^2 = r^2$, and $g_{zz} = h_z^2 = 1$. Therefore, the contravariant metric tensor reads $g^{rr} = 1$, $g^{\theta\theta} = r^{-2}$, and $g^{zz} = 1$. Using Equations 10.20 and 10.25 and considering that indices i, j, k, m are various permutations of cylindrical variables r,θ,z we receive the Christoffel symbol of the second kind as

$$\left\{\begin{aligned}
\Gamma^r &= \begin{bmatrix} \Gamma^r_{rr} & \Gamma^r_{r\theta} & \Gamma^r_{rz} \\ \Gamma^r_{\theta r} & \Gamma^r_{\theta\theta} & \Gamma^r_{\theta z} \\ \Gamma^r_{zr} & \Gamma^r_{z\theta} & \Gamma^r_{zz} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\Gamma^\theta &= \begin{bmatrix} \Gamma^\theta_{rr} & \Gamma^\theta_{r\theta} & \Gamma^\theta_{rz} \\ \Gamma^\theta_{\theta r} & \Gamma^\theta_{\theta\theta} & \Gamma^\theta_{\theta z} \\ \Gamma^\theta_{zr} & \Gamma^\theta_{z\theta} & \Gamma^\theta_{zz} \end{bmatrix} = \begin{bmatrix} 0 & 1/r & 0 \\ 1/r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\Gamma^z &= \begin{bmatrix} \Gamma^z_{rr} & \Gamma^z_{r\theta} & \Gamma^z_{rz} \\ \Gamma^z_{\theta r} & \Gamma^z_{\theta\theta} & \Gamma^z_{\theta z} \\ \Gamma^z_{zr} & \Gamma^z_{z\theta} & \Gamma^z_{zz} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}\right. \qquad 10.27$$

For example, $\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^\theta_{r\theta} = \frac{1}{h_\theta}\frac{\partial(h_\theta)}{\partial r} = \frac{1}{r}\frac{\partial r}{\partial r} = 1/r$ and $\Gamma^r_{\theta\theta} = -\frac{h_\theta}{h_r^2}\frac{\partial(h_\theta)}{\partial r} = -r\frac{\partial r}{\partial r} = -r$. Recall that the symbols, given by Equation 10.27 are defined based on the covariant basis vectors, $\vec{e}_i$ (see Equation 10.5), or $\frac{\partial \vec{e}_r}{\partial \theta} = \underbrace{(1/r)}_{\Gamma^\theta_{r\theta}}\vec{e}_\theta$ and $\frac{\partial \vec{e}_\theta}{\partial \theta} = \underbrace{(-r)}_{\Gamma^r_{\theta\theta}}\vec{e}_r$. It is useful to write the Christoffel symbols in terms of the physical components of the vectors involved. This is done by using Equation 3.8 (i.e. $\vec{e}_i = h_i\vec{e}(i)$), or $\frac{\partial(r\vec{e}(\theta))}{\partial \theta} = -r\vec{e}(r)$, hence $\frac{\partial(\vec{e}(\theta))}{\partial \theta} = -\vec{e}(r)$, or the Christoffel symbol based on unit vector reads $\hat{\Gamma}^r_{\theta\theta} = -1$ (hatted to distinguish them from the relation given in Equation 10.27). Similarly, for $\frac{\partial \vec{e}_r}{\partial \theta} = \underbrace{(1/r)}_{\Gamma^\theta_{r\theta}}\vec{e}_\theta$ we can write $\frac{\partial(\vec{e}(r))}{\partial \theta} = \frac{1}{r}(r\vec{e}(\theta))$, hence $\frac{\partial(\vec{e}(r))}{\partial \theta} = \vec{e}(\theta)$, or the Christoffel symbol based on unit vector reads $\hat{\Gamma}^\theta_{r\theta} = 1$.

We could also calculate the Christoffel symbols $\hat{\Gamma}^\theta_{r\theta}$ and $\hat{\Gamma}^r_{\theta\theta}$, using the unit vectors or the physical components, as given in Example 8.1. Writing these relations in vector form, we have $\vec{e}(r) = \begin{Bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{Bmatrix}$, $\vec{e}(\theta) = \begin{Bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{Bmatrix}$,

and $\vec{e}(z)=\begin{Bmatrix}0\\0\\1\end{Bmatrix}$. Therefore, the derivatives are; $\frac{\partial\vec{e}(r)}{\partial r}=\frac{\partial\vec{e}(\theta)}{\partial r}=\frac{\partial\vec{e}(z)}{\partial r}=0$,
$\frac{\partial\vec{e}(r)}{\partial\theta}=\begin{Bmatrix}-\sin\theta\\\cos\theta\\0\end{Bmatrix}=\vec{e}(\theta)$, $\quad\frac{\partial\vec{e}(\theta)}{\partial\theta}=\begin{Bmatrix}-\cos\theta\\-\sin\theta\\0\end{Bmatrix}=-\vec{e}(r)$, $\frac{\partial\vec{e}(z)}{\partial\theta}=0$, $\quad\frac{\partial\vec{e}(r)}{\partial z}=$
$\frac{\partial\vec{e}(\theta)}{\partial z}=\frac{\partial\vec{e}(z)}{\partial z}=0$. Therefore, the non-zero Christoffel symbols calculated based on the unit vectors are $\hat{\Gamma}^{\theta}_{r\theta}=1$ and $\hat{\Gamma}^{r}_{\theta\theta}=-1$. The final results can be summarized as

$$\begin{cases}\hat{\Gamma}^{r}=\begin{bmatrix}\hat{\Gamma}^{r}_{rr}&\hat{\Gamma}^{r}_{r\theta}&\hat{\Gamma}^{r}_{rz}\\\hat{\Gamma}^{r}_{\theta r}&\hat{\Gamma}^{r}_{\theta\theta}&\hat{\Gamma}^{r}_{\theta z}\\\hat{\Gamma}^{r}_{zr}&\hat{\Gamma}^{r}_{z\theta}&\hat{\Gamma}^{r}_{zz}\end{bmatrix}=\begin{bmatrix}0&0&0\\0&-1&0\\0&0&0\end{bmatrix}\\\hat{\Gamma}^{\theta}=\begin{bmatrix}\hat{\Gamma}^{\theta}_{rr}&\hat{\Gamma}^{\theta}_{r\theta}&\hat{\Gamma}^{\theta}_{rz}\\\hat{\Gamma}^{\theta}_{\theta r}&\hat{\Gamma}^{\theta}_{\theta\theta}&\hat{\Gamma}^{\theta}_{\theta z}\\\hat{\Gamma}^{\theta}_{zr}&\hat{\Gamma}^{\theta}_{z\theta}&\hat{\Gamma}^{\theta}_{zz}\end{bmatrix}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}\\\hat{\Gamma}^{z}=\begin{bmatrix}\hat{\Gamma}^{z}_{rr}&\hat{\Gamma}^{z}_{r\theta}&\hat{\Gamma}^{z}_{rz}\\\hat{\Gamma}^{z}_{\theta r}&\hat{\Gamma}^{z}_{\theta\theta}&\hat{\Gamma}^{z}_{\theta z}\\\hat{\Gamma}^{z}_{zr}&\hat{\Gamma}^{z}_{z\theta}&\hat{\Gamma}^{z}_{zz}\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\end{cases}\tag{10.28}$$

Spherical coordinates

From the results of Example 8.2, for a spherical system (r,φ,θ), (note the order of the coordinates defined here, see Figure 8.3) we can write the non-zero terms of the metric tensor as $g_{rr}=h_r^2=1$, $g_{\varphi\varphi}=h_\varphi^2=r^2$, and $g_{\theta\theta}=h_\theta^2=r^2\sin^2\varphi$. Therefore, the contravariant metric tensor reads $g^{rr}=1$, $g^{\varphi\varphi}=1/r^2$, and $g^{\theta\theta}=1/\left(r^2\sin^2\varphi\right)$. Using Equations 10.20 and 10.25 and considering that indices i,j,k,m are various permutations of spherical variables r,φ,θ we get the Christoffel symbol of the second kind as

$$\begin{cases} \Gamma^r = \begin{bmatrix} \Gamma^r_{rr} & \Gamma^r_{r\varphi} & \Gamma^r_{r\theta} \\ \Gamma^r_{\varphi r} & \Gamma^r_{\varphi\varphi} & \Gamma^r_{\varphi\theta} \\ \Gamma^r_{\theta r} & \Gamma^r_{\theta\varphi} & \Gamma^r_{\theta\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & -r\sin^2\varphi \end{bmatrix} \\ \Gamma^\varphi = \begin{bmatrix} \Gamma^\varphi_{rr} & \Gamma^\varphi_{r\varphi} & \Gamma^\varphi_{r\theta} \\ \Gamma^\varphi_{\varphi r} & \Gamma^\varphi_{\varphi\varphi} & \Gamma^\varphi_{\varphi\theta} \\ \Gamma^\varphi_{\theta r} & \Gamma^\varphi_{\theta\varphi} & \Gamma^\varphi_{\theta\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1/r & 0 \\ 1/r & 0 & 0 \\ 0 & 0 & -\dfrac{\sin 2\varphi}{2} \end{bmatrix} \\ \Gamma^\theta = \begin{bmatrix} \Gamma^\theta_{rr} & \Gamma^\theta_{r\varphi} & \Gamma^\theta_{r\theta} \\ \Gamma^\theta_{\varphi r} & \Gamma^\theta_{\varphi\varphi} & \Gamma^\theta_{\varphi\theta} \\ \Gamma^\theta_{\theta r} & \Gamma^\theta_{\theta\varphi} & \Gamma^\theta_{\theta\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1/r \\ 0 & 0 & \cot\varphi \\ 1/r & \cot\varphi & 0 \end{bmatrix} \end{cases} \qquad 10.29$$

For example, $\Gamma^2_{33} = \Gamma^\varphi_{\theta\theta} = -\frac{h_3}{h_2^2}\frac{\partial h_3}{\partial x^2} = -\frac{r\sin\varphi}{r^2}\frac{\partial(r\sin\varphi)}{\partial\varphi} = -\sin\varphi\cos\varphi = -\frac{\sin 2\varphi}{2}$. Recall that the symbols are defined based on the covariant basis vectors, $\vec{e}_i$ (see Equation 10.5), or $\frac{\partial \vec{e}_\theta}{\partial\theta} = \underbrace{\left(-\frac{\sin 2\varphi}{2}\right)}_{\Gamma^\varphi_{\theta\theta}}\vec{e}_\varphi$.

We can write this relation in terms of the physical components, as $\frac{\partial\left(r\sin\varphi\vec{e}(\theta)\right)}{\partial\theta} = \left(-\frac{\sin 2\varphi}{2}\right)r\vec{e}(\varphi)$, or $\frac{\partial\left(\vec{e}(\theta)\right)}{\partial\theta} = \underbrace{(-\cos\varphi)}_{\hat{\Gamma}^\varphi_{\theta\theta}}\vec{e}(\varphi)$. Similarly, the other non-zero symbols can be written as; $\Gamma^r_{\varphi\varphi} = -\frac{h_\varphi}{h_r^2}\frac{\partial h_\varphi}{\partial r} = -r\frac{\partial r}{\partial r} = -r$, $\Gamma^r_{\theta\theta} = -\frac{h_\theta}{h_r^2}\frac{\partial h_\theta}{\partial r} = -r\sin\varphi\frac{\partial(r\sin\varphi)}{\partial r} = -r\sin^2\varphi$, $\Gamma^\varphi_{\varphi r} = \Gamma^\varphi_{r\varphi} = \frac{1}{h_\varphi}\frac{\partial h_\varphi}{\partial r} = \frac{1}{r}\frac{\partial r}{\partial r} = \frac{1}{r}$, $\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{h_\theta}\frac{\partial h_\theta}{\partial r} = \frac{1}{r\sin\varphi}\frac{\partial(r\sin\varphi)}{\partial r} = \frac{1}{r}$, and $\Gamma^\theta_{\varphi\theta} = \Gamma^\theta_{\theta\varphi} = \frac{1}{h_\theta}\frac{\partial h_\theta}{\partial\varphi} = \frac{1}{r\sin\varphi}\frac{\partial(r\sin\varphi)}{\partial\varphi} = \cot\varphi$. In terms of physical components of the corresponding vectors, we have $\frac{\partial\vec{e}_\varphi}{\partial\varphi} = \underbrace{(-r)}_{\Gamma^r_{\varphi\varphi}}\vec{e}_r$ or $\frac{\partial(r\vec{e}(\varphi))}{\partial\varphi} = -r\vec{e}(r) = -\vec{e}(r)$, hence $\hat{\Gamma}^r_{\varphi\varphi} = -1$. Also, $\frac{\partial\vec{e}_\theta}{\partial\theta} = \underbrace{\left(-r\sin^2\varphi\right)}_{\Gamma^r_{\theta\theta}}\vec{e}_r$

or $\dfrac{\partial\left(r\sin\varphi\vec{e}(\theta)\right)}{\partial\theta}=-r\sin^2\varphi\vec{e}(r)=\underbrace{(-\sin\varphi)}_{\hat{\Gamma}^{r}_{\theta\theta}}\vec{e}(r)$, hence $\hat{\Gamma}^{r}_{\theta\theta}=-\sin\varphi$.

Also, $\dfrac{\partial\vec{e}_r}{\partial\varphi}=\underbrace{(1/r)}_{\Gamma^{\varphi}_{r\varphi}}\vec{e}_\varphi$ or $\dfrac{\partial\vec{e}(r)}{\partial\varphi}=(1/r)r\vec{e}(\varphi)=\vec{e}(\varphi)$, hence $\hat{\Gamma}^{\varphi}_{r\varphi}=1$. Also, $\dfrac{\partial\vec{e}_r}{\partial\theta}=\underbrace{(1/r)}_{\Gamma^{\theta}_{r\theta}}\vec{e}_\theta$ or $\dfrac{\partial\vec{e}(r)}{\partial\theta}=(1/r)\left(r\sin\varphi\vec{e}(\theta)\right)=\underbrace{\sin\varphi}_{\hat{\Gamma}^{\theta}_{r\theta}}\vec{e}(\theta)$, hence $\hat{\Gamma}^{\theta}_{r\theta}=\sin\varphi$.

Also, $\dfrac{\partial\vec{e}_\varphi}{\partial\theta}=\underbrace{(\cot\varphi)}_{\Gamma^{\theta}_{\varphi\theta}}\vec{e}_\theta$ or $\dfrac{\partial\left(r\vec{e}(\varphi)\right)}{\partial\theta}=\cot\varphi\left(r\sin\varphi\vec{e}(\theta)\right)=\underbrace{\cos\varphi}_{\hat{\Gamma}^{\theta}_{\varphi\theta}}\vec{e}(\theta)$, hence $\hat{\Gamma}^{\theta}_{\varphi\theta}=\cos\varphi$.

The Christoffel symbols can be obtained directly using the unit vectors as well. From the results of Example 8.1, writing the physical components relations in vector form, we have $\vec{e}(r)=\begin{Bmatrix}\sin\varphi\cos\theta\\ \sin\varphi\sin\theta\\ \cos\varphi\end{Bmatrix}$, $\vec{e}(\varphi)=\begin{Bmatrix}\cos\varphi\cos\theta\\ \cos\varphi\sin\theta\\ -\sin\varphi\end{Bmatrix}$, and $\vec{e}(\theta)=\begin{Bmatrix}-\sin\theta\\ \cos\theta\\ 0\end{Bmatrix}$. Therefore, the derivatives are; $\dfrac{\partial\vec{e}(r)}{\partial r}=\dfrac{\partial\vec{e}(\varphi)}{\partial r}=\dfrac{\partial\vec{e}(\theta)}{\partial r}=0$,

$$\frac{\partial\vec{e}(r)}{\partial\varphi}=\begin{Bmatrix}\cos\varphi\cos\theta\\ \cos\varphi\sin\theta\\ -\sin\varphi\end{Bmatrix}=\vec{e}(\varphi),\qquad \frac{\partial\vec{e}(\varphi)}{\partial\varphi}=\begin{Bmatrix}-\sin\varphi\cos\theta\\ -\sin\varphi\sin\theta\\ -\cos\varphi\end{Bmatrix}=-\vec{e}(r),\ \frac{\partial\vec{e}(\theta)}{\partial\varphi}=0,$$

$$\frac{\partial\vec{e}(r)}{\partial\theta}=\begin{Bmatrix}-\sin\varphi\sin\theta\\ \sin\varphi\cos\theta\\ 0\end{Bmatrix}=\vec{e}(\theta)\sin\varphi,\qquad \frac{\partial\vec{e}(\varphi)}{\partial\theta}=\begin{Bmatrix}-\cos\varphi\sin\theta\\ \cos\varphi\cos\theta\\ 0\end{Bmatrix}=\vec{e}(\theta)\cos\varphi,$$

$\dfrac{\partial\vec{e}(\theta)}{\partial\theta}=\begin{Bmatrix}-\cos\theta\\ -\sin\theta\\ 0\end{Bmatrix}=-\vec{e}(r)\sin\varphi-\vec{e}(\varphi)\cos\varphi$. Therefore, the non-zero Christoffel symbols calculated based on the unit vectors are

$$\begin{cases} \hat{\Gamma}^r = \begin{bmatrix} \hat{\Gamma}^r_{rr} & \hat{\Gamma}^r_{r\varphi} & \hat{\Gamma}^r_{r\theta} \\ \hat{\Gamma}^r_{\varphi r} & \hat{\Gamma}^r_{\varphi\varphi} & \hat{\Gamma}^r_{\varphi\theta} \\ \hat{\Gamma}^r_{\theta r} & \hat{\Gamma}^r_{\theta\varphi} & \hat{\Gamma}^r_{\theta\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\sin\varphi \end{bmatrix} \\ \hat{\Gamma}^\varphi = \begin{bmatrix} \hat{\Gamma}^\varphi_{rr} & \hat{\Gamma}^\varphi_{r\varphi} & \hat{\Gamma}^\varphi_{r\theta} \\ \hat{\Gamma}^\varphi_{\varphi r} & \hat{\Gamma}^\varphi_{\varphi\varphi} & \hat{\Gamma}^\varphi_{\varphi\theta} \\ \hat{\Gamma}^\varphi_{\theta r} & \hat{\Gamma}^\varphi_{\theta\varphi} & \hat{\Gamma}^\varphi_{\theta\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\cos\varphi \end{bmatrix} \\ \hat{\Gamma}^\theta = \begin{bmatrix} \hat{\Gamma}^\theta_{rr} & \hat{\Gamma}^\theta_{r\varphi} & \hat{\Gamma}^\theta_{r\theta} \\ \hat{\Gamma}^\theta_{\varphi r} & \hat{\Gamma}^\theta_{\varphi\varphi} & \hat{\Gamma}^\theta_{\varphi\theta} \\ \hat{\Gamma}^\theta_{\theta r} & \hat{\Gamma}^\theta_{\theta\varphi} & \hat{\Gamma}^\theta_{\theta\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \sin\varphi \\ 0 & 0 & \cos\varphi \\ \sin\varphi & \cos\varphi & 0 \end{bmatrix} \end{cases} \qquad 10.30$$

Readers can use the Christoffel symbols of the 2nd kind calculated in this example and further calculate the Christoffel symbols of the 1st kind using Equation 10.19.

CHAPTER 11

DERIVATIVE FORMS—CURL, DIVERGENCE, LAPLACIAN

In the governing equations for physical phenomena we usually have terms which contain various forms of gradient operator, including the gradient vector itself—for example, when the gradient vector operates as cross product or dot product with another tensor quantity. The cross-product is called the *curl*, $\vec{\nabla} \times \vec{A}$ and the dot-product is the *divergence*, $\vec{\nabla} \cdot \vec{A}$. For example, in fluid mechanics the curl of the velocity vector is the vorticity vector and divergence of velocity vector vanishes, for incompressible fluids.

As mentioned, one of the objectives of tensor analysis is to provide a tool to write down the governing equations in coordinate-independent forms while the covariancy or contravariancy of tensor quantities are correctly implemented in these equations. In the following sections, we derive the coordinate-independent relations for curl, divergence, Laplacian, and biharmonic operators of tensors for an arbitrary coordinate system.

11.1 CURL OPERATIONS ON TENSORS

To form the curl operator expression, we first consider it for a vector in 3D coordinate space and then extend it to higher order dimension. In Section 9, we gave a list of guidelines for forming expressions for tensors. Following these guidelines and using Equation 9.6, we can write the covariant component of the curl of a vector as $C_i = \left(\vec{\nabla} \times \vec{A}\right)_i = \mathcal{E}_{ijk} \nabla^j A^k$. We know that $\mathcal{E}_{ijk}$ is a tensor (see Section 9). But the expression $\nabla^j A^k$ does not transform like a tensor, due to extra terms appearing when transformed to another arbitrary coordinate system [4], [7]. However, if we use the results obtained in Section 10 and replace $\nabla^j A^k$ with the tensor $A^{k\,j}_{,\cdot}$ (i.e., the contravariant derivative of the contravariant component) then we get the proper expression as (see Equation 10.8)

$$C_i = \mathcal{E}_{ijk} A^{k\,j}_{,\cdot} \tag{11.1}$$

Similarly, the contravariant component of the curl is written as, using Equations 9.6 and 10.12,

$$C^i = \mathcal{E}^{ijk} A_{k,j} \tag{11.2}$$

Both expressions given by Equations 11.1 and 11.2 are tensors, since terms are involved are tensors as well as both reduce to the Cartesian forms when written in the Cartesian coordinate system (i.e., $C_i = e_{ijk} \dfrac{\partial A_k}{\partial y^j}$ because the covariant and contravariant components are identical and Jacobian is unity).

To have a more practical and explicit relation for the curl, we write Equation 11.2 in detail or $C^i = \mathcal{E}^{ijk} A_{k,j} = \mathcal{E}^{ijk} \dfrac{\partial A_k}{\partial x^j} - \mathcal{E}^{ijk} \Gamma^n_{jk} A_n$ examining the last term, on the R.H.S, using the symmetry property of the Christoffel symbol (i.e., $\Gamma^n_{jk} = \Gamma^n_{kj}$) and the anti-symmetry of the permutation symbol (i.e., $\mathcal{E}^{ijk} = -\mathcal{E}^{ikj}$) yields, $\mathcal{E}^{ijk}\Gamma^n_{jk} A_n = -\mathcal{E}^{ikj}\Gamma^n_{kj} A_n$. But after interchanging $j \leftrightarrow k$, dummy indices, only in the term on the R.H.S of the latter expression we receive $-\mathcal{E}^{ikj}\Gamma^n_{kj} A_n = -\mathcal{E}^{ijk}\Gamma^n_{jk} A_n$. After comparing this result with the original expression (i.e., $-\mathcal{E}^{ijk}\Gamma^n_{jk} A_n = \mathcal{E}^{ijk}\Gamma^n_{jk} A_n$) we can conclude that $\mathcal{E}^{ijk}\Gamma^n_{jk} A_n = 0$. Therefore, Equation 11.2 can be written as

$$C^i = \left(\vec{\nabla} \times \vec{A}\right)^i = \frac{e_{ijk}}{\mathcal{J}} \left(\frac{\partial A_k}{\partial x^j}\right) \tag{11.3}$$

The curl vector is then written as $\vec{C} = C^i \vec{e}_i$.

Similarly, we write Equation 11.1 in detail, also using Equation 10.8, or $C_i = \mathcal{E}_{ijk} A^{k\,j}_{,\cdot} = \mathcal{E}_{ijk} g^{jn} \dfrac{\partial A^k}{\partial x^n} + \mathcal{E}_{ijk} g^{jn} \Gamma^k_{nm} A^m$. Examining the last term on the R.H.S, it turns out that it doesn't vanish. Therefore, we have

$$C_i = \left(\vec{\nabla} \times \vec{A}\right)_i = \mathcal{J} g^{jn} e_{ijk} \left(\frac{\partial A^k}{\partial x^n} + \Gamma^k_{nm} A^m \right) \qquad 11.4$$

Note that the curl operation result is a tensor of the same rank as the original quantities. As in our example $\vec{C}$ is a vector, like $\vec{\nabla}$ and $\vec{A}$.

Now, we extend the discussion and find the relation for tensors of higher rank. For simplicity of writing the expressions, without losing generality, we consider a mixed tensor of second rank A^i_j. The invariant quantity is $A = A^i_j \vec{e}_i \vec{e}^{\,j}$ and the curl reads $\vec{\nabla} \times A = \vec{\nabla} \times \left(A^i_j \vec{e}_i \vec{e}^{\,j} \right)$. The gradient operator just differentiates all the terms in front of it but the vector part of $\vec{\nabla}$ can form a cross-product with either $\vec{e}_i$ or $\vec{e}^{\,j}$, both are possible and legitimate operations. One can form, then two forms of the curl of the quantity A. In general, we can have N number of forms for the curl of a tensor of order N. For our example here, we have two forms:

1. We consider the case for which the cross-product operation occurring with $\vec{e}_i$. Therefore, the contravariant component of A^i_j should be involved with the permutation tensor indices. We can write, using Equation 11.1, the component of curl of A as $\left(\vec{\nabla} \times \vec{e}_i\right) A^i_j \vec{e}^{\,j}$. Careful attention should be given to pick up the right indices to form the curl expression. Since we picked the $\vec{e}_i$ for cross-product operation with gradient, the covariant permutation tensor should be used, say $\mathcal{E}_{kmn}$. Therefore, the index m corresponds to the contravariant differentiation and index n summed up with the contravariant component of the tensor A. The index k, is then the covariant component of the result or the curl tensor. The rank of the result should be the same as the original quantity A^i_j (i.e., rank two), hence the free covariant index j remains intact and we end up with a doubly-covariant tensor, or

$$C_{kj} = \mathcal{E}_{kmn} A^{n\,m}_{j,\,\cdot} = \mathcal{J} g^{mp} e_{kmn} \left(\frac{\partial A^n_j}{\partial x^p} + \Gamma^n_{pq} A^q_j \right) \qquad 11.5$$

2. We consider the case for which the cross-product operation occurs with $\vec{e}^{\,j}$. Therefore, the covariant component of A^i_j should be involved with the permutation tensor. We can write, using Equation 11.1, the component of curl of A as $\left(\vec{\nabla} \times \vec{e}^{\,j}\right) A^i_j \vec{e}_i$. Careful attentions should be given to pick up the right indices to form the curl expression. Since we picked the $\vec{e}^{\,j}$ for cross-product operation with gradient, the contravariant

permutation tensor should be used, say $\mathcal{E}^{kmn}$. Therefore, the index m corresponds to the covariant differentiation and index n is summed up with the covariant component of the tensor A. The index k, is then the contravariant component of the result or the curl. But the rank of the result should be the same as the original quantity A^i_j (i.e., rank two), hence the free contravariant index i remains intact and we end up with a doubly contravariant tensor, or

$$C^{ki} = \mathcal{E}^{kmn} A^i_{n,m} = \frac{e_{kmn}}{\mathcal{J}} \left(\frac{\partial A^i_n}{\partial x^m} \right) \qquad 11.6$$

A similar operation can be used to write down the curl of a tensor of rank N, which has N possible outcomes.

11.2 PHYSICAL COMPONENTS OF THE CURL OF TENSORS—3D ORTHOGONAL SYSTEMS

In many applications, we consider orthogonal systems (i.e., $g_{ij} = 0$, $i \neq j$). In this section, we derive expressions for the physical components of the curl of a vector $\vec{A}$ in 3D orthogonal systems. Recalling from previous sections (see Sections 3 and 8), we can write $\vec{A} = A^i \vec{e}_i = A_i \vec{e}^{\,i} = A(i)\vec{e}(i)$ and $h_i = |\vec{e}_i|$. Using $\vec{e}_i \cdot \vec{e}^{\,j} = \delta^j_i$, for orthogonal systems we have $\vec{e}_i \cdot \vec{e}^{\,i} = 1$ which means that the covariant basis vector $\vec{e}_i$ and contravariant basis vector $\vec{e}^{\,i}$ both point in the same direction. For example, $\vec{e}^{\,1}$ and $\vec{e}_1$ are along the same line and point in the same direction. Therefore,

$$|\vec{e}^{\,i}| = 1/h_i \qquad 11.7$$

Substituting $\vec{e}_i = h_i \vec{e}(i)$ in $A(i)\vec{e}(i) = A^i \vec{e}_i$ and $\vec{e}^{\,i} = \vec{e}(i)/h_i$ in $A(i)\vec{e}(i) = A_i \vec{e}^{\,i}$ gives

$$A(i) = h_i A^i = A_i / h_i \qquad \textit{no sum on } i \qquad 11.8$$

Or $A(1) = h_1 A^1 = A_1 / h_1$, $A(2) = h_2 A^2 = A_2 / h_2$, $A(3) = h_3 A^3 = A_3 / h_3$.

Now we would like to write the physical components of the curl of vector $\vec{A}$. To do this we multiply Equation 11.3 by h_i, or $\left(\vec{\nabla}\times\vec{A}\right)^i h_i = h_i C^i = h_i \frac{e_{ijk}}{\mathcal{J}}\left(\frac{\partial A_k}{\partial x^j}\right)$. But $h_i C^i = C(i)$ and $A_k = h_k A(k)$ (see Equation 11.8). Therefore, we will get the physical component $C(i)$ of the curl of vector $\vec{A}$ in terms of its physical component $A(k)$, as

$$C(i) = h_i \frac{e_{ijk}}{\mathcal{J}} \frac{\partial\left(h_k A(k)\right)}{\partial x^j} \qquad \textit{no sum on i, orthogonal} \qquad 11.9$$

Or, using $\mathcal{J} = h_1 h_2 h_3$ for orthogonal systems, we have

$$\begin{cases} \left(\vec{\nabla}\times\vec{A}\right)(1) = C(1) = \dfrac{1}{h_2 h_3}\left[\dfrac{\partial\left(h_3 A(3)\right)}{\partial x^2} - \dfrac{\partial\left(h_2 A(2)\right)}{\partial x^3}\right] \\ \left(\vec{\nabla}\times\vec{A}\right)(2) = C(2) = \dfrac{1}{h_1 h_3}\left[\dfrac{\partial\left(h_1 A(1)\right)}{\partial x^3} - \dfrac{\partial\left(h_3 A(3)\right)}{\partial x^1}\right] \\ \left(\vec{\nabla}\times\vec{A}\right)(3) = C(3) = \dfrac{1}{h_1 h_2}\left[\dfrac{\partial\left(h_2 A(2)\right)}{\partial x^1} - \dfrac{\partial\left(h_1 A(1)\right)}{\partial x^2}\right] \end{cases} \qquad 11.10$$

It can be concluded that for a Cartesian system we recover the familiar expressions, or $\left(\vec{\nabla}\times\vec{A}\right)_1 = C_1 = \left(\frac{\partial A_z}{\partial Y} - \frac{\partial A_y}{\partial Z}\right)$, $\left(\vec{\nabla}\times\vec{A}\right)_2 = C_2 = \left(\frac{\partial A_X}{\partial Z} - \frac{\partial A_Z}{\partial X}\right)$, and $\left(\vec{\nabla}\times\vec{A}\right)_3 = C_3 = \left(\frac{\partial A_Y}{\partial X} - \frac{\partial A_X}{\partial Y}\right)$, since $h_1 = h_2 = h_3 = 1$.

11.3 DIVERGENCE OPERATION ON TENSORS

To form the divergence operator expression, we first consider it for a vector in 3D coordinate space and then extend the results to higher order dimension. In Section 9, we gave a list of guidelines for forming expressions. Following these guidelines and using Equation 9.2, we can write the divergence (using the covariant component of the gradient) as

$$\vec{\nabla}\cdot\vec{A} = A^i_{,i} \qquad 11.11$$

which reduces to proper expression in Cartesian coordinates and shows that the divergence of a vector results in a scalar, or in general divergence operation reduces the rank of the tensor, on which it is operating, by one. To further simplify Equation 11.11, we write it explicitly (see Equation 10.6) as $A^i_{,i} = \frac{\partial A^i}{\partial x^i} + \Gamma^i_{in} A^n$ and plug in for Γ^i_{in} from Equation 10.26 (after interchanging dummy indices $i \leftrightarrow n$), to get $A^i_{,i} = \frac{\partial A^i}{\partial x^i} + \frac{A^i}{\mathcal{J}}\frac{\partial \mathcal{J}}{\partial x^i}$. After substituting for $A^i_{,i}$ into Equation 11.11 and performing some manipulations, we receive

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{\mathcal{J}} \frac{\partial\left(\mathcal{J} A^i\right)}{\partial x^i} \tag{11.12}$$

In addition, Equation 11.11 can be written in terms of the covariant component of the vector using g^{ij}, or $\vec{\nabla} \cdot \vec{A} = \left(g^{ij} A_j\right)_{,i}$, after expanding gives $\vec{\nabla} \cdot \vec{A} = g^{ij}\left(A_j\right)_{,i} + A_j \underbrace{\left(g^{ij}\right)_{,i}}_{=0} = g^{ij} A_{j,i}$ or after interchanging $i \leftrightarrow j$ and using the symmetry of metric tensor, we receive

$$\vec{\nabla} \cdot \vec{A} = g^{ij} A_{i,j} \tag{11.13}$$

Now we extend the discussion by considering a tensor of higher rank, for example and without losing generality a tensor of second rank $A = A^i_j \vec{e}_i \vec{e}^{\,j}$. The divergence of A is $\vec{\nabla} \cdot A = \vec{\nabla} \cdot \left(A^i_j \vec{e}_i \vec{e}^{\,j}\right)$ in which we have choices between $\vec{e}_i$ or $\vec{e}^{\,j}$ for the gradient vector performing a dot-product operation. Therefore, we have two cases for this example:

1. $\vec{\nabla}$ dot products with $\vec{e}_i$: In this case we get $\vec{\nabla} \cdot A = \vec{\nabla} \cdot \vec{e}_i \left(A^i_j \vec{e}^{\,j}\right)$. Note that the gradient differentiates all terms in front of it but as a vector onlydotted with $\vec{e}_i$. In other words, the contravariant index i is involved in the dot-product operation. Using Equation 11.11 we can write $\vec{\nabla} \cdot A = A^i_{j,i}$ which is a tensor of rank one, as expected since divergence reduces the rank of A^i_j by one. But using Equation 11.12, we have $A^i_{j,i} = \frac{\partial A^i_j}{\partial x^i} + \Gamma^i_{im} A^m_j - \Gamma^m_{ij} A^i_m$ in which the first two terms on the R.H.S. gives $\frac{\partial A^i_j}{\partial x^i} + \Gamma^i_{im} A^m_j = \frac{1}{\mathcal{J}} \frac{\partial\left(\mathcal{J} A^i_j\right)}{\partial x^i}$, using Equation 11.12. Finally, we get the result as

$$\vec{\nabla} \cdot A = \vec{\nabla} \cdot \vec{e}_i \left(A^i_j \vec{e}^{\,j}\right) = \frac{1}{\mathcal{J}} \frac{\partial\left(\mathcal{J} A^i_j\right)}{\partial x^i} - \Gamma^m_{ij} A^i_m \tag{11.14}$$

2. $\vec{\nabla}$ dot products with $\vec{e}^{\,j}$: In this case we get $\vec{\nabla}\cdot A = \vec{\nabla}\cdot\vec{e}^{\,j}\left(A^i_j\vec{e}_i\right)$. Note that the gradient differentiates all terms in front of it but as a vector only dotted with $\vec{e}^{\,j}$. In other words, the index j is involved in the dot-product operation. Using Equation 11.11 we can write $\vec{\nabla}\cdot A = g^{kj}A^i_{k,j}$ which is a tensor of rank one, as expected since divergence reduces the rank of A^i_j by one. Therefore, we have

$$\vec{\nabla}\cdot A = \vec{\nabla}\cdot\vec{e}^{\,j}\left(A^i_j\vec{e}_i\right) = A^{i\ \ k}_{k,\ .} \tag{11.15}$$

Using similar operations, we can write the expression for divergence of a tensor of rank N. Both relations given by Equations 11.14 and 11.15 reduce to familiar forms for divergence in a Cartesian coordinate system. For example, for divergence of vector $\vec{A}$ we get $\vec{\nabla}\cdot\vec{A} = A_{i,i} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$.

11.4 LAPLACIAN OPERATIONS ON TENSORS

A Laplacian operator is the result of a gradient operator forming a dot-product with itself, or divergence of gradient, $\nabla^2 = \left(\vec{\nabla}\cdot\vec{\nabla}\right)$. Performing Laplacian on a scalar ψ and using Equation 11.12, we can write $\nabla^2\psi = \vec{\nabla}\cdot\left(\vec{\nabla}\psi\right) = \left(\nabla\psi\right)^i_{,i}$, or $\nabla^2\psi = \frac{1}{\mathcal{J}}\frac{\partial\left(\mathcal{J}\nabla^i\psi\right)}{\partial x^i}$ where $\nabla^i\psi = g^{ij}\frac{\partial\psi}{\partial x^j}$. Therefore,

$$\nabla^2\psi = \frac{1}{\mathcal{J}}\frac{\partial}{\partial x^i}\left(\mathcal{J}g^{ij}\frac{\partial\psi}{\partial x^j}\right) \tag{11.16}$$

Since ∇^2 is a tensor of rank zero, then $\nabla^2\psi$ is a scalar, as well. In general, the rank is determined by the quantity that Laplacian is operating on.

We extend the discussion to find the Laplacian of a vector, for example $\vec{A} = A^i\vec{e}_i$. Since for Laplacian the dot-product operation is performed between the two gradients involved, we don't have many choices, as was the case for the divergence operator, and hence the formulation is more definite in terms of the results. Using Equations 11.13 and 11.16, we can write the contravariant component of the gradient of vector $\vec{A}$ as $g^{nm}A^i_{,n}$ and the components of the Laplacian as $\left(g^{nm}A^i_{,n}\right)_{,m}$. Note that index m indicates the dot-product operation between the contravariant component of the gradient of $\vec{A}$ and the covariant derivative of the second gradient involved. Expanding the last expression, we receive $\left(g^{nm}A^i_{,n}\right)_{,m} = g^{nm}_{,m}A^i_{,n} + g^{nm}A^i_{,nm}$. But $g^{nm}_{,m} = 0$, since it is a tensor and its value is zero in the Cartesian coordinate system, hence it should be zero for an arbitrary system, as well. Therefore, we have

$$\left(\nabla^2\vec{A}\right)^i = \left(g^{nm}A^i_{,n}\right)_{,m} = g^{nm}A^i_{,nm} \qquad 11.17$$

The term $A^i_{,nm}$ which is the second covariant derivative is a new term, so far. By expanding this term, we get $A^i_{,nm} = \left(A^i_{,n}\right)_{,m} = \dfrac{\partial\left(A^i_{,n}\right)}{\partial x^m} + \Gamma^i_{mj}A^j_{,n} - \Gamma^j_{mn}A^i_{,j}$. But $A^i_{,n} = \dfrac{\partial\left(A^i\right)}{\partial x^n}A^i + \Gamma^i_{nk}A^k$ and after substituting and rearranging similar terms, we receive

$$\begin{aligned} A^i_{,nm} &= \frac{\partial^2 A^i}{\partial x^m \partial x^n} + A^k\frac{\partial \Gamma^i_{nk}}{\partial x^m} + \Gamma^i_{nk}\frac{\partial A^k}{\partial x^m} + \Gamma^i_{mj}\frac{\partial A^i}{\partial x^j} \\ &\quad - \Gamma^j_{mn}\frac{\partial A^i}{\partial x^j} + \Gamma^i_{mj}\Gamma^j_{nk}A^k - \Gamma^j_{mn}\Gamma^i_{jk}A^k \end{aligned} \qquad 11.18$$

Similar expressions can be written for higher order tensors, using hints from Equation 11.18 for writing the terms involving Christoffel symbols with appropriate signs. The higher the rank of $\vec{A}$ the more complicated the related expression for $\nabla^2\vec{A}$ becomes.

For an orthogonal coordinate system, Equation 11.17 simplifies, since $g^{nm} = 0$ for $m \neq n$, or (for $N = 3$)

$$\left(\nabla^2\vec{A}\right)^i = g^{11}A^i_{,11} + g^{22}A^i_{,22} + g^{33}A^i_{,33} \quad (orthogonal)$$

11.5 BIHARMONIC OPERATIONS ON TENSORS

A biharmonic operator is the result of a Laplacian operator operating on itself, or $\nabla^4 = \nabla^2\left(\nabla^2\right)$. Performing the operation on a scalar ψ and using Equation 11.12, we can write

$$\nabla^4\psi = \frac{1}{\mathcal{J}}\frac{\partial}{\partial x^k}\left\{\mathcal{J}g^{kl}\frac{\partial}{\partial x^l}\underbrace{\left[\frac{1}{\mathcal{J}}\frac{\partial}{\partial x^i}\left(\mathcal{J}g^{ij}\frac{\partial\psi}{\partial x^j}\right)\right]}_{\nabla^2\psi}\right\} \qquad 11.19$$

When a biharmonic operates on a vector, the result is a vector as well. For example, for $\vec{A} = A^i \vec{e}_i$, we can write $\nabla^4 \vec{A} = B^i \vec{e}_i$. Since the result is a vector we can use Equation 11.17 to write

$$B^i = g^{lm} g^{jk} A^i_{,jklm} \tag{11.20}$$

The quantity $A^i_{,jklm}$ is a lengthy expression and can be written in detail [2], [4], [7] using hints taken from Equation 11.18. In principle, one can extend the discussion to find operators like $\nabla^{2n} = \underbrace{\nabla^2 \nabla^2 \cdots \nabla^2}_{n\ times}$. In practice we encounter mostly up to the level of the biharmonic operator in governing equations.

11.6 PHYSICAL COMPONENTS OF THE LAPLACIAN OF A VECTOR—3D ORTHOGONAL SYSTEMS

In this section, we derive expressions for physical components of the Laplacian of a vector. Relations for physical components are useful since they are the magnitude of the components using unit vectors as the scale of measurement (see Section 6).

We start with Equation 11.18 and write it for a vector, like $\vec{A} = A(i)\vec{e}(i)$ or $\nabla^2 \vec{A} = \dfrac{1}{\mathcal{J}} \dfrac{\partial}{\partial x^i}\left(\mathcal{J} g^{ij} \dfrac{\partial \vec{A}}{\partial x^j} \right)$. In an orthogonal system, we have $g^{ij} = 0$ (when $i \neq j$) and $g^{ii} = 1/h_i^2$. Therefore, after substitution into the expression, we will have $\nabla^2 \vec{A} = \dfrac{1}{\mathcal{J}} \dfrac{\partial}{\partial x^i}\left(\dfrac{\mathcal{J}}{h_i^2} \dfrac{\partial \left[A(i)\vec{e}(i) \right]}{\partial x^i} \right)$. Performing the differentiations along with using related relations for Christoffel symbols and after some manipulations, we get the physical j-component as [7]

$$\begin{aligned}
\nabla^2 \vec{A}\Big|_{(j)} &= \nabla^2 A(j) + 2\left[\frac{1}{h_i h_j^2} \frac{\partial h_j}{\partial x^i} \frac{\partial A(i)}{\partial x^j} - \frac{1}{h_j h_i^2} \frac{\partial h_i}{\partial x^j} \frac{\partial A(i)}{\partial x^i} \right] \\
&\quad + \left[\frac{A(i)}{h_i h_j^3} \frac{\partial h_j}{\partial x^j} \frac{\partial h_j}{\partial x^i} + \frac{A(i)}{h_j h_i^3} \frac{\partial h_i}{\partial x^i} \frac{\partial h_i}{\partial x^j} \right] - \left[\frac{A(j)}{h_i^2 h_j^2} \frac{\partial h_j}{\partial x^i} \frac{\partial h_j}{\partial x^i} + \frac{A(k)}{h_k h_j h_i^2} \frac{\partial h_i}{\partial x^k} \frac{\partial h_i}{\partial x^j} \right] \\
&\quad + \left[\frac{A(i)}{\mathcal{J}} \frac{\partial}{\partial x^j}\left(\frac{\mathcal{J}}{h_i h_j^2} \frac{\partial h_j}{\partial x^i} \right) - \frac{A(i)}{\mathcal{J}} \frac{\partial}{\partial x^i}\left(\frac{\mathcal{J}}{h_j h_i^2} \frac{\partial h_i}{\partial x^j} \right) \right]
\end{aligned} \tag{11.21}$$

In Equation 11.21, summation is done on indices i and k but not on index j, since it is the free index. To show the application of Equation 11.21, we present two examples to calculate the components of Laplacian of a vector in cylindrical and spherical coordinate systems. Note that the Laplacian of a vector is a vector as well, which has three components.

11.6.1 Example: Physical components of the Laplacian of a vector—cylindrical systems

Consider a cylindrical coordinate system $\left(x^1,x^2,x^3\right)\equiv(r,\theta,z)$ where r is the radial distance, θ the azimuthal angle, and z the vertical coordinate, w.r.t. the Cartesian coordinates $\left(y^1,y^2,y^3\right)\equiv(X,Y,Z)$. The functional relations are $\begin{cases} X = r\cos\theta \\ Y = r\sin\theta \\ Z = z \end{cases}$. Find the Laplacian expressions for a scalar and the physical components of a vector in this system. Let $A(1)=A_r$, $A(2)=A_\theta$, and $A(3)=A_z$.

Solution:

To find the covariant basis vectors, we use $\vec{e}_i=\dfrac{\partial y^j}{\partial x^i}\vec{E}_j$. Therefore, we get $\vec{e}_1=\dfrac{\partial X}{\partial r}\vec{E}_1+\dfrac{\partial Y}{\partial r}\vec{E}_2+\dfrac{\partial Z}{\partial r}\vec{E}_3=\cos\theta\vec{E}_1+\sin\theta\vec{E}_2$. Similarly, $\vec{e}_2=\dfrac{\partial X}{\partial\theta}\vec{E}_1+\dfrac{\partial Y}{\partial\theta}\vec{E}_2+\dfrac{\partial Z}{\partial\theta}\vec{E}_3=-r\sin\theta\vec{E}_1+r\cos\theta\vec{E}_2$. And $\vec{e}_3=\dfrac{\partial X}{\partial z}\vec{E}_1+\dfrac{\partial Y}{\partial z}\vec{E}_2+\dfrac{\partial Z}{\partial z}\vec{E}_3=\vec{E}_3$. Therefore, we have

$$\begin{cases} \vec{e}_r=\cos\theta\vec{E}_1+\sin\theta\vec{E}_2 \\ \vec{e}_\theta=-r\sin\theta\vec{E}_1+r\cos\theta\vec{E}_2 \\ \vec{e}_z=\vec{E}_3 \end{cases}$$

The scale factors are the magnitudes of the basis vectors. Hence $h_1=h_r=\sqrt{\vec{e}_r\cdot\vec{e}_r}=1$, $h_2=h_\theta=\sqrt{\vec{e}_\theta\cdot\vec{e}_\theta}=r$, and $h_3=h_z=\sqrt{\vec{e}_z\cdot\vec{e}_z}=1$. The unit vectors $\vec{e}(i)=\dfrac{\vec{e}_i}{h_i}$, are

$$\begin{cases} \vec{e}(r)=\cos\theta\vec{E}_1+\sin\theta\vec{E}_2 \\ \vec{e}(\theta)=-\sin\theta\vec{E}_1+\cos\theta\vec{E}_2 \\ \vec{e}(\varphi)=\vec{E}_3 \end{cases}$$

The Jacobin is $\mathcal{J} = h_r h_\theta h_z = r$ and $g^{ii} = h_i^{-2}$, or $g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Now using Equation 11.16 for orthogonal system i.e. $\nabla^2 \psi = \frac{1}{\mathcal{J}} \frac{\partial}{\partial x^i}\left(\mathcal{J} g^{ii} \frac{\partial \psi}{\partial x^i} \right)$, since $g^{ij} = 0 \text{ for } i \neq j$, and substituting for corresponding values we will get $\nabla^2 \psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2}$. In other words, the Laplacian operator in cylindrical coordinate system reads

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2}\left(\frac{\partial^2}{\partial \theta^2} \right) + \frac{\partial^2}{\partial z^2} \qquad 11.22$$

Similarly, using Equation 11.21 we get the physical components of the Laplacian for the vector $\vec{A}$, as

$$\begin{cases} \nabla^2 \vec{A}\Big|_r = \nabla^2 A_r - \frac{2}{r^2}\frac{\partial A_\theta}{\partial \theta} - \frac{A_r}{r^2} \\ \nabla^2 \vec{A}\Big|_\theta = \nabla^2 A_\theta + \frac{2}{r^2}\frac{\partial A_r}{\partial \theta} - \frac{A_\theta}{r^2} \\ \nabla^2 \vec{A}\Big|_z = \nabla^2 A_z \end{cases} \qquad 11.23$$

Where the operator ∇^2 is given by Equation 11.22 and A_r, A_θ, A_z represent the physical components of $\vec{A}$.

11.6.2 Example: Physical components of the Laplacian of a vector—spherical systems

Consider a spherical coordinate system $\left(x^1, x^2, x^3\right) \equiv \left(r, \varphi, \theta\right)$ where r is the radial distance, φ the polar and θ the azimuthal angle, w.r.t. the Cartesian coordinates $\left(y^1, y^2, y^3\right) \equiv (X, Y, Z)$. The functional relations are

$\begin{cases} X = r \sin\varphi \cos\theta \\ Y = r \sin\varphi \sin\theta \\ Z = r\cos\varphi \end{cases}$. Find the Laplacian expressions for a scalar and the physical components of a vector in this system. Let $A(1) = A_r$, $A(2) = A_\varphi$, and $A(3) = A_\theta$.

Solution:

To find the covariant basis vectors, we use $\vec{e}_i = \frac{\partial y^j}{\partial x^i}\vec{E}_j$. Therefore, we get $\vec{e}_1 = \frac{\partial X}{\partial r}\vec{E}_1 + \frac{\partial Y}{\partial r}\vec{E}_2 + \frac{\partial Z}{\partial r}\vec{E}_3 = \sin\varphi\cos\theta\vec{E}_1 + \sin\varphi\sin\theta\vec{E}_2 + \cos\varphi\vec{E}_3$. Similarly, $\vec{e}_2 = \frac{\partial X}{\partial \varphi}\vec{E}_1 + \frac{\partial Y}{\partial \varphi}\vec{E}_2 + \frac{\partial Z}{\partial \varphi}\vec{E}_3 = r\cos\varphi\cos\theta\vec{E}_1 + r\cos\varphi\sin\theta\vec{E}_2 - r\sin\varphi\vec{E}_3$. And $\vec{e}_3 = \frac{\partial X}{\partial \theta}\vec{E}_1 + \frac{\partial Y}{\partial \theta}\vec{E}_2 + \frac{\partial Z}{\partial \theta}\vec{E}_3 = -r\sin\varphi\sin\theta\vec{E}_1 + r\sin\varphi\cos\theta\vec{E}_2$. Therefore, we have

$$\begin{cases} \vec{e}_r = \sin\varphi\cos\theta\vec{E}_1 + \sin\varphi\sin\theta\vec{E}_2 + \cos\varphi\vec{E}_3 \\ \vec{e}_\varphi = r\cos\varphi\cos\theta\vec{E}_1 + r\cos\varphi\sin\theta\vec{E}_2 - r\sin\varphi\vec{E}_3 \\ \vec{e}_\theta = -r\sin\varphi\sin\theta\vec{E}_1 + r\sin\varphi\cos\theta\vec{E}_2 \end{cases}$$

The scale factors are the magnitudes of the basis vectors. Hence $h_1 = h_r = \sqrt{\vec{e}_r \cdot \vec{e}_r} = 1$, $h_2 = h_\varphi = \sqrt{\vec{e}_\varphi \cdot \vec{e}_\varphi} = r$, and $h_3 = h_\theta = \sqrt{\vec{e}_\theta \cdot \vec{e}_\theta} = r\sin\varphi$. The unit vectors $\vec{e}(i) = \frac{\vec{e}_i}{h_i}$, are

$$\begin{cases} \vec{e}(r) = \sin\varphi\cos\theta\vec{E}_1 + \sin\varphi\sin\theta\vec{E}_2 + \cos\varphi\vec{E}_3 \\ \vec{e}(\varphi) = \cos\varphi\cos\theta\vec{E}_1 + \cos\varphi\sin\theta\vec{E}_2 - \sin\varphi\vec{E}_3 \\ \vec{e}(\varphi) = -\sin\theta\vec{E}_1 + \cos\theta\vec{E}_2 \end{cases}$$

The Jacobin is $\mathcal{J} = h_r h_\varphi h_\theta = r^2\sin\varphi$ and $g^{ii} = h_i^{-2}$, or $g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/\left(r^2\sin^2\varphi\right) \end{bmatrix}$. Now using Equation 11.16 for orthogonal system i.e. $\nabla^2\psi = \frac{1}{\mathcal{J}}\frac{\partial}{\partial x^i}\left(\mathcal{J}g^{ii}\frac{\partial\psi}{\partial x^i}\right)$, since $g^{ij} = 0$ for $i \neq j$, and substituting for corresponding values we will get $\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial\psi}{\partial\varphi}\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2\psi}{\partial\theta^2}$. In other words, the Laplacian operator in spherical coordinate system reads

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2}{\partial\theta^2} \qquad 11.24$$

Similarly, using Equation 11.21 we receive the physical components of the Laplacian for the vector $\vec{A}$, as

$$\begin{cases} \nabla^2\vec{A}\Big|_r = \nabla^2 A_r - \dfrac{2A_r}{r^2} - \dfrac{2}{r^2\sin\varphi}\dfrac{\partial(\sin\varphi A_\varphi)}{\partial\varphi} - \dfrac{2}{r^2\sin\varphi}\dfrac{\partial A_\theta}{\partial\theta} \\ \nabla^2\vec{A}\Big|_\varphi = \nabla^2 A_\varphi - \dfrac{A_\varphi}{r^2\sin^2\varphi} + \dfrac{2}{r^2}\dfrac{\partial A_r}{\partial\varphi} - \dfrac{2\cos\varphi}{r^2\sin^2\varphi}\dfrac{\partial A_\theta}{\partial\theta} \\ \nabla^2\vec{A}\Big|_\theta = \nabla^2 A_\theta - \dfrac{A_\theta}{r^2\sin^2\varphi} + \dfrac{2}{r^2\sin\varphi}\dfrac{\partial A_r}{\partial\theta} + \dfrac{2\cos\varphi}{r^2\sin^2\varphi}\dfrac{\partial A_\varphi}{\partial\theta} \end{cases} \qquad 11.25$$

Where the operator ∇^2 is given by Equation 11.24 and A_r, A_φ, A_θ represent the physical components of $\vec{A}$.

In the next section, we focus on relations pertinent to tensor transformations between merely Cartesian coordinate systems.

CHAPTER 12

CARTESIAN TENSOR TRANSFORMATION—ROTATIONS

When transformation of tensors is performed from one Cartesian coordinate system to another one, relations derived in previous sections take simpler forms, called *Cartesian tensors*. As mentioned previously, a Cartesian system consists of three mutually perpendicular flat surfaces for which; the basis vectors are unit vectors, there is no distinction between covariant and contravariant components, and the components of a tensor are the physical components. Therefore, we use subscript indices for all types of components, regardless.

We will consider two Cartesian systems, y_i and y'_i with a common origin—otherwise we can always redefine them to have a common origin with transforming, say, the origin of y'_i to that of y_i. Following the discussion from Section 3, we can write the differential displacement $\overrightarrow{ds}$ in these systems, as

$$\overrightarrow{ds} = dy_i \vec{E}_i = dy'_j \vec{E}'_j \qquad 12.1$$

where $\vec{E}_i$ and $\vec{E}'_j$ are the corresponding unit vectors. However, we have $dy_i = \dfrac{\partial y_i}{\partial y'_j} dy'_j$, and after substituting in Equation 12.1 we get

$\left(\frac{\partial y_i}{\partial y'_j}\vec{E}_i - \vec{E}'_j\right) dy'_j = 0$. Since dy'_j is arbitrary, we conclude $\frac{\partial y_i}{\partial y'_j}\vec{E}_i = \vec{E}'_j$ or, by $i \leftrightarrow j$, we have

$$\vec{E}'_i = \frac{\partial y_j}{\partial y'_i}\vec{E}_j \tag{12.2}$$

Now performing dot-product on both sides of Equation 12.2 with $\vec{E}_k$ gives, $\vec{E}_k \cdot \vec{E}'_i = \frac{\partial y_j}{\partial y'_i}\underbrace{\vec{E}_j \cdot \vec{E}_k}_{\delta_{jk}} = \frac{\partial y_k}{\partial y'_i}$. Hence,

$$\vec{E}_k \cdot \vec{E}'_i = \frac{\partial y_k}{\partial y'_i} \tag{12.3}$$

This means that the cosine of the angle between y_k and y'_i is equal to $\frac{\partial y_k}{\partial y'_i}$, called the *cosine direction*.

Similarly, we can write $\vec{E}_k = \frac{\partial y'_i}{\partial y_k}\vec{E}'_i$, and after dot-product with $\vec{E}'_j$ we receive $\vec{E}'_j \cdot \vec{E}_k = \frac{\partial y'_i}{\partial y_k}\underbrace{\vec{E}'_i \cdot \vec{E}'_j}_{\delta_{ij}} = \frac{\partial y'_j}{\partial y_k}$. Hence, interchanging $i \leftrightarrow j$, we receive

$$\vec{E}'_i \cdot \vec{E}_k = \frac{\partial y'_i}{\partial y_k} \tag{12.4}$$

This means that the cosine of the angle between y_k and y'_i is equal to $\frac{\partial y'_i}{\partial y_k}$. Therefore, comparing Equations 12.3 and 12.4 and considering the commutativity of the dot-product operation, we will have

$$\frac{\partial y'_i}{\partial y_k} = \frac{\partial y_k}{\partial y'_i} \tag{12.5}$$

Equation 12.5 yields $dy_i = \frac{\partial y_i}{\partial y'_j}\partial y'_j = \frac{\partial y'_j}{\partial y_i}dy'_j$ and $\vec{E}'_i = \frac{\partial y_j}{\partial y'_i}\vec{E}_j = \frac{\partial y'_i}{\partial y_j}\vec{E}_j$, but only in Cartesian coordinate systems. In other words, these relations prove that there is no distinction between covariant and contravariant components in a Cartesian system (see Equations 4.8 and 5.4, for comparison).

12.1 ROTATION MATRIX

Now if we assume that coordinate system y'_i is obtained by rotating coordinates y_j at an angle θ about an axis parallel to a unit vector $\vec{n} = (n_1, n_2, n_3)$, then the quantity $\dfrac{\partial y'_i}{\partial y_j} = R_{ij}$ is a function of θ and $\vec{n}$ and it can be shown that [9], it reads as

$$R_{ij} = n_i n_j + \left(\delta_{ij} - n_i n_j\right)\cos\theta + e_{ijk} n_k \sin\theta \qquad 12.6$$

Equation 12.6 is *Rodrigues' rotation formula*, or the 3D rotation matrix. Readers should note that the Right-Hand-Rule convention applies to the rotation of coordinates with respect to the positive direction of unit vector $\vec{n}$ when using Equation 12.6. For example, for $\vec{n} = (0,0,1)$ and $\theta = \pi/2$, R_{ij} gives the transformation of $x - y$ plane about the positive direction of z-axis for an angle of $90°$.

Now we integrate $dy'_i = \dfrac{\partial y'_i}{\partial y_j} dy_j = R_{ij} dy_j$ which yields, $y'_i = R_{ij} y_j$. Note that the constant of integration is zero due to having a common origin for both systems, and R_{ij} is a constant for given $\vec{n}$ and θ. Similarly, we can rotate y'_i at an angle $(-\theta)$ about the same unit vector $\vec{n}$ to recover the original y_j coordinates. Therefore, we can write $y_i = R^{-1}_{ij} y'_j = R_{ji} y'_j$, by having R^{-1}_{ij} being equal to the inverse or transpose of R_{ij} [4]. Further examining Equation 12.6, we can conclude that R_{ij} for a given angle θ is equal to R_{ji}, or its transpose, calculated for angle $(-\theta)$. This is the result of the property of the last term on the R.H.S of Equation 12.6 when $i \leftrightarrow j$ to get R_{ji}, or $e_{ijk} n_k \sin\theta = -e_{jik} n_k \sin\theta = e_{jik} n_k \sin(-\theta)$. The remaining terms are not affected by this index-change operation.

12.2 EQUIVALENT SINGLE ROTATION: EIGENVALUES AND EIGENVECTORS

It would be useful to write the matrix form of the Cartesian coordinate rotations, as well. For example (in a 3D space with $N = 3$), $y'_i = R_{ij} y_j$ can be written as $\begin{Bmatrix} y'_1 \\ y'_2 \\ y'_3 \end{Bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix}$, and after using Equation 12.6, for $\vec{n} = (n_1, n_2, n_3)$ and rotation angle θ, we get

$$\begin{Bmatrix} y'_1 \\ y'_2 \\ y'_3 \end{Bmatrix} = \begin{bmatrix} \cos\theta + n_1 n_1 (1-\cos\theta) & n_1 n_2 (1-\cos\theta) + n_3 \sin\theta & n_1 n_3 (1-\cos\theta) - n_2 \sin\theta \\ n_1 n_2 (1-\cos\theta) - n_3 \sin\theta & \cos\theta + n_2 n_2 (1-\cos\theta) & n_2 n_3 (1-\cos\theta) + n_1 \sin\theta \\ n_1 n_3 (1-\cos\theta) + n_2 \sin\theta & n_2 n_3 (1-\cos\theta) - n_1 \sin\theta & \cos\theta + n_3 n_3 (1-\cos\theta) \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} \qquad 12.7$$

From Equation 12.7 (or 12.6) we can conclude that the trace of the rotation matrix (i.e., the sum of diagonal elements, R_{ii}) is equal to $1 + 2\cos\theta$, (note that $n_i n_i = |\vec{n}|^2 = n_1^2 + n_2^2 + n_3^2 = 1$, since $\vec{n}$ is a unit vector), or

$$\theta = \cos^{-1}\left(\frac{R_{ii} - 1}{2}\right) \qquad 12.8$$

Equation 12.8 is a useful relation for finding rotation angle θ, once R_{ij} is known/given. However, readers may ask: if R_{ij} is known, how can one find the corresponding unit vector parallel to the axis of rotation? This question is all the more significant considering that we can have a sequence of rotations to arrive at a final desired orientation of the coordinate axes. For example, if we rotate the original system through a sequence of rotations and arrive at the final system orientation, i.e., $y_i \to \underbrace{y_i^{t1} \to \cdots \to y_i^{tn}}_{sequence} \to y'_i$, then we can apply Equation 12.7 for each sequence and calculate the final rotation matrix. That is $\{y'_i\} = \underbrace{\left[R_{ij}^{tn}\right]\left[R_{ij}^{tn-1}\right]\cdots\left[R_{ij}^{t1}\right]}_{[R_{ij}]}\{y_i\}$. Note that for each rotation in the sequence, the corresponding rotation matrix is pre-multiplied to the previous ones, and the normal vector parallel to the rotation axis is defined based on the current coordinate system at hand in the sequence (see Example 12.2.1)

Now, if we want to replace all intermediate rotations with just one equivalent rotation, Equation 12.8 gives the value of the angle for the desired equivalent single rotation. However, to find the axis of rotation parallel to the unit vector for the equivalent single rotation we need a mathematical procedure/tool. This tool can be obtained by finding the real eigenvector of the given rotation matrix since, through all rotations, only the eigenvectors remain in the same direction [2], [4]. The mathematical procedure is as follows:

We consider a normal unit vector n_i perpendicular to a plane and write the components of the rotation tensor on this plane, or $R_{ij}n_j$. This quantity is a vector whose components are the elements of the rotation tensor on the plane considered. We identify this quantity as $R_{ij}n_j = T_i$. In general, the direction of the vector T_i does not necessarily coincide with the normal vector to the plane, as shown in Figure 12.1.

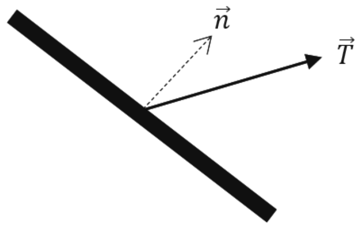


FIGURE 12.1 Rotation matrix component $\vec{T}$, about an arbitrary plane with normal $\vec{n}$.

In other words, we could demand a specific plane and calculate its corresponding normal direction such that vector T_i ends up in the same direction as the normal to the plane considered. Therefore, we can write $T_i n_i = R_{ij}n_j n_i$. This quantity, which is the projection of T_i in the direction of n_i, is a quadratic function of vector n_i. To calculate its extremum value with the constraint that $n_i n_i = 1$ (which means that having the normal vector as a unit vector), we use the Lagrange multiplier method for extremizing the scalar quantity $\mathcal{M} = R_{ij}n_j n_i - \lambda(n_i n_i - 1)$, where the constant λ is the Lagrange multiplier, or the eigenvalue of R_{ij}. This is achieved by letting $\dfrac{\partial \mathcal{M}}{\partial n_i} = 0$, which leads to (note that the $\dfrac{\partial \mathcal{M}}{\partial \lambda} = 0$ recovers the constraint, $n_i n_i = 1$)

$$\left(R_{ij} - \lambda\delta_{ij}\right)n_j = 0 \qquad 12.9$$

In matrix form, Equation 12.9 can be written as $\begin{bmatrix} R_{11}-\lambda & R_{12} & R_{13} \\ R_{21} & R_{22}-\lambda & \varepsilon_{23} \\ R_{31} & R_{32} & R_{33}-\lambda \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$. This system of equations has a non-trivial (i.e., non-zero) solution, for n_i, if the determinant of the matrix is zero, or

$$\det\left(R_{ij} - \lambda\delta_{ij}\right) = \begin{vmatrix} R_{11}-\lambda & R_{12} & R_{13} \\ R_{21} & R_{22}-\lambda & \varepsilon_{23} \\ R_{31} & R_{32} & R_{33}-\lambda \end{vmatrix} = 0 \qquad 12.10$$

By evaluating the determinant, we get a cubic equation in terms of λ (which has three roots, or eigenvalues) as

$$\lambda^3 - \underbrace{\left(R_{ii}\right)}_{\mathbb{I}_1}\lambda^2 + \underbrace{\left(\frac{R_{ii}R_{jj} - R_{ij}R_{ji}}{2}\right)}_{\mathbb{I}_2}\lambda - \underbrace{\left|R_{ij}\right|}_{\mathbb{I}_3} = 0 \qquad 12.11$$

Equation 12.11 is known as the *characteristic equation* for matrix R_{ij}, where $\mathbb{I}_1$ is the first *principal scalar invariant,* which is equal to the trace of R_{ij}; or $\mathbb{I}_1 = R_{ii} = \lambda_1 + \lambda_2 + \lambda_3$, $\mathbb{I}_2$ is the second *principal scalar invariant,* which is $\mathbb{I}_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$, or the sum of the diagonal minors; and $\mathbb{I}_3$ is the third *principal scalar invariant,* which is equal to the determinant of R_{ij}, or $\mathbb{I}_3 = e_{ijk}R_{1i}R_{2j}R_{3k} = \lambda_1\lambda_2\lambda_3$.

Equation 12.11 contains several important properties and pieces of information about the system's rotation matrix. Some examples related to the discussion here include:

- For symmetric tensors with real components there exist three real value answers for λ (or λ_1, λ_2, and λ_3). This is not the case for R_{ij}, since it is not symmetric, and we get only one real value λ, [10].
- The answers for λ are the *eigenvalues* (or principal values) of R_{ij}.
- The corresponding n_i, calculated and normalized for each eigenvalue, are the *eigenvectors* (or principal directions) of R_{ij}.
- A given eigenvector when multiplied by a constant real number (positive or negative) it gives a new vector, but all have the same direction parallel to the principal direction.

The method of finding equivalent rotation is very useful in practice for designing the motion of machine parts in robotic applications, such as a robot arm's motion.

12.2.1 Example: Equivalent single rotation to sequential rotations of a Cartesian system

Having a Cartesian system labeled as $y_i = (y_1, y_2, y_3)$ and its transformation to another system labeled as $y'_i = (y'_1, y'_2, y'_3)$ such that y'_1 coincides with y_2, y'_2 with y_3, and y'_3 with y_1, as shown in Figure 12.2, find the overall equivalent single rotation matrix for $y'_i = R_{ij}y_j$, the axis of rotation direction $\vec{n}$ and the angle of rotation θ.

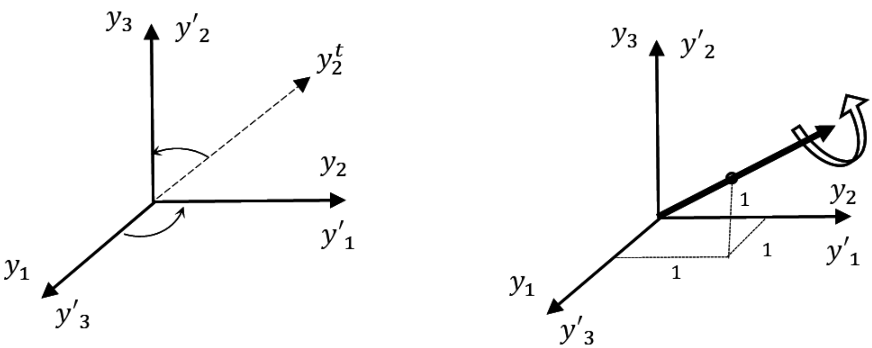


FIGURE 12.2 Sequential rotations of a Cartesian coordinate system (left) and the equivalent single rotation (right).

Solution:

We consider two sequential rotations: 1) rotation about y_3-axis with normal vector $(0,0,1)$ with an angle of 90° (R.H.R. applies) to get the intermediate transformation $(y_1, y_2, y_3) \rightarrow (y'_1, y^t_2, y_3)$, and 2) rotation about new y'_1 with normal vector $(1,0,0)$ with an angle of 90° (R.H.R. applies) to get the second transformation $(y'_1, y^t_2, y_3) \rightarrow (y'_1, y'_2, y'_3)$ (see Figure 12.2). Therefore, using Equation 12.7 and the data given for the first transformation with $\vec{n} = (0,0,1)$ we have $\begin{Bmatrix} y'_1 \\ y^t_2 \\ y_3 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix}$. Similarly, for the second rotation with $\vec{n} = (1,0,0)$, we have $\begin{Bmatrix} y'_1 \\ y'_2 \\ y'_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{Bmatrix} y'_1 \\ y^t_2 \\ y_3 \end{Bmatrix}$. Note that the second rotation is performed on the current system at hand, which is the result of the first rotation. Therefore, after substitution we can write $\begin{Bmatrix} y'_1 \\ y'_2 \\ y'_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix}$, or $\begin{Bmatrix} y'_1 \\ y'_2 \\ y'_3 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix}$. The final rotation matrix is then $R_{ij} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Note that each rotation matrix pre-multiplies the previous one in the sequence of rotations. Having the final rotation matrix, we can calculate the single rotation angle. Using Equation 12.8, we get $\theta = \cos^{-1}\left(\frac{R_{ii}-1}{2}\right) = \cos^{-1}\left(\frac{0-1}{2}\right) = \frac{2\pi}{3}$. The corresponding rotation axis is the eigenvector associated with the real eigenvalue for $R_{ij} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. We can write $\left|R_{ij} - \lambda\delta_{ij}\right| = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{bmatrix} = -\lambda^3 + 1 = 0$. Hence, we get the characteristic equation as $(\lambda-1)(\lambda^2+\lambda+1) = 0$ with its only real root being $\lambda = 1$. Note that we could use Equation 12.11 instead for calculating the eigenvalues. For $\lambda = 1$, we get the eigenvector by solving $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = (1)\begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}$, or $\begin{cases} n_2 = n_1 \\ n_3 = n_2 \\ n_1 = n_3 \end{cases}$. This system is not determinate but gives $n_1 = n_2 = n_3$. However, the constraint for having $\vec{n}$ as a unit vector gives $n_1^2 + n_2^2 + n_3^2 = 1$, or $3n_1^2 = 1$. Therefore, $n_1 = n_2 = n_3 = \frac{\sqrt{3}}{3}$ and finally we have the equivalent rotation axis or the unit vector $\vec{n} = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$.

In conclusion, we can say that instead of two intermediate sequential rotations, as mentioned above, we can rotate the original $y_i = (y_1, y_2, y_3)$ at an angle of $120°$, using right-hand-rule convention, about an axis parallel to the unit vector $\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$ (see Figure 12.2). Readers should note that for calculating the rotation matrix using Equation 12.7, values of the components of $\vec{n}$ should be those of the unit vector. However, to obtain the direction of the rotation axis we can multiply the components of the unit vector by a constant—for this example the multiplier is $\sqrt{3}$. As shown in Figure 12.2, the result is a vector, or (1,1,1), which is a vector along the same direction as the unit vector $\vec{n}$. This is consistent with the property of the eigenvalues. We can examine these results by using Equation 12.6 to calculate the single rotation matrix, using $\theta = 120°$ and $\vec{n} = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$, or $R_{11} = \left(\frac{\sqrt{3}}{3}\right)^2 + \left(1 - \left(\frac{\sqrt{3}}{3}\right)^2\right)\cos(120°) = \frac{1}{3} - \frac{1}{3} = 0$, similarly $R_{22} = R_{33} = 0$.

Also, we get $R_{12} = \left(\frac{\sqrt{3}}{3}\right)^2 + \left(\underbrace{\delta_{12}}_{=0} - \left(\frac{\sqrt{3}}{3}\right)^2\right)\cos(120°) + \underbrace{e_{123}}_{=1}\left(\frac{\sqrt{3}}{3}\right)\sin(120°) = 1.$

Similarly, we receive $R_{23} = R_{31} = 1$, and the remaining elements $R_{21} = R_{32} = R_{13} = 0$. These results are identical with the one obtained using sequential rotations.

CHAPTER 13

COORDINATE INDEPENDENT GOVERNING EQUATIONS

Reliable mathematical models, also referred to as governing equations, are important tools for engineering analysis. A reliable and validated mathematical model of a physical phenomenon is a set of algebraic relations among quantities and their various derivatives, such as Newton's 2^{nd} law of motion, equilibrium equations for momentum flux, Fourier's law of heat flux, Fick's law of mass flux, Navier-Stokes equations for flow of fluids, Maxwell's electromagnetic equations, etc. These governing equations, along with some fundamental principles (like the 2^{nd} law of thermodynamics, conservations of energy, mass, electric charge, etc.) form the foundation of engineering science and its applications.

The quantities related to any physical phenomenon can be represented by tensors of different ranks stated in the relevant governing equations, such as force and acceleration vectors and mass of a body as a scalar, in Newton's 2^{nd} law of motion; gradient of temperature in Fourier's law; divergence of velocity vector in fluid flow, etc. Sometimes, for analysis and design purposes, we need to have the relevant governing equations written in coordinate systems other than the Cartesian system. For example, for analyzing mechanical stresses in the wall of a cylindrical pressure vessel we prefer to choose the cylindrical polar coordinate system that is a natural fit to the shape of the vessel. This requirement has encouraged scientists and engineers to define various coordinate systems suitable for solving the governing equations in practice. In other words, and in the context of tensor analysis, we would like to use the relations derived and discussed in the previous sections to write down the terms involved in well-known governing equations in engineering in a form that is general enough for application in an arbitrary but well-defined coordinate system.

In this section, we derive some new relations mostly used in engineering, in addition to those discussed in the previous sections. We hope that this helps readers in writing down similar coordinate independent terms involved in equations of their choice for their applications.

13.1 THE ACCELERATION VECTOR—CONTRAVARIANT COMPONENTS

Newton's 2nd law is a mathematical model for the motion mechanics of physical objects—specifically, it is the balance of applied forces and the rate of change of momentum, or mass times acceleration. Velocity is the time rate of change of displacement vector $\overrightarrow{ds}$ (see Section 2). Therefore, in an arbitrary coordinate system x^i, we can write the velocity vector $\vec{v}$ as $\vec{v} = \frac{\overrightarrow{ds}}{dt}$ for the corresponding time increment of dt. Using Equation 3.6, we have $\vec{v} = \frac{\overrightarrow{ds}}{dt} = \vec{e}_i \frac{dx^i}{dt}$. Since $\overrightarrow{ds}$ is a vector, hence $\vec{v}$ is a vector and we can write it in terms of its contravariant component v^i, or

$$\vec{v} = v^i \vec{e}_i = \frac{dx^i}{dt} \vec{e}_i \qquad 13.1$$

Equation 13.1 clearly gives the contravariant component of the velocity vector as $v^i = \frac{dx^i}{dt}$. Acceleration vector $\vec{a}$ is the time derivative of the velocity vector, or

$$\vec{a} = \frac{d\vec{v}}{dt} = a^i \vec{e}_i \qquad 13.2$$

Using Equation 13.1, we can write $\frac{d\vec{v}}{dt} = \frac{d\left(v^i \vec{e}_i\right)}{dt} = \vec{e}_i \frac{dv^i}{dt} + v^i \frac{d\vec{e}_i}{dt}$. To relate the time derivative to the space gradient we use chain rule $\frac{d}{dt} = \underbrace{\frac{dx^j}{dt}}_{v^j} \frac{\partial}{\partial x^j}$, hence

$$\frac{d}{dt} = v^j \frac{\partial}{\partial x^j} \qquad 13.3$$

Substituting into Equation 13.2, we get $\vec{a} = \frac{d\vec{v}}{dt} = \vec{e}_i \left(v^j \frac{\partial v^i}{\partial x^j} \right) + v^i \left(v^j \frac{\partial \vec{e}_i}{\partial x^j} \right)$.

But $\frac{\partial \vec{e}_i}{\partial x^j} = \Gamma^k_{ij} \vec{e}_k$, using Equation 10.5. Therefore, $\vec{a} = \vec{e}_i v^j \frac{\partial v^i}{\partial x^j} + v^i v^j \Gamma^k_{ij} \vec{e}_k$ and in the last term we interchange dummy indices $i \leftrightarrow k$ and use the symmetry property of the Christoffel symbol to obtain

$$\vec{a} = \left(v^j \frac{\partial v^i}{\partial x^j} + v^k v^j \Gamma^i_{jk} \right) \vec{e}_i \qquad 13.4$$

Comparing Equations 13.2 and 13.4, we get the contravariant component of the acceleration vector as

$$a^i = v^j \left(\frac{\partial v^i}{\partial x^j} + v^k \Gamma^i_{jk} \right) \qquad 13.5$$

In addition, we can write $v^i_{,j} = \frac{\partial v^i}{\partial x^j} + v^k \Gamma^i_{jk}$, after using Equation 10.6, to receive the compact form of Equation 13.5, as

$$a^i = v^j v^i_{,j} \qquad 13.6$$

Equation 13.6 recovers the familiar expression for acceleration in Cartesian coordinates (i.e., $a^i = v^j \frac{\partial v^i}{\partial x^j}$).

We can also write the covariant component of the acceleration vector using a metric tensor, or $a_k = g_{ik} a^i$ (see Section 8) for an arbitrary coordinate system.

Now, we can write the Newton's 2nd law in a general coordinate independent form. We choose to use the contravariant component of the acceleration vector; hence, for compatibility we must use the contravariant component of the total force vector, $\vec{F} = F^i \vec{e}_i$, as well. Therefore, we can write

$$F^i = m v^j \frac{\partial v^i}{\partial x^j} + m v^k v^j \Gamma^i_{jk} = m v^j v^i_{,j} \qquad 13.7$$

For example, in a 3D coordinate system we get three equations of motions for each dimension, with summation on indices j and k only, as

$$\begin{cases} F^1 = mv^j \dfrac{\partial v^1}{\partial x^j} + mv^k v^j \Gamma^1_{jk} \\ F^2 = mv^j \dfrac{\partial v^2}{\partial x^j} + mv^k v^j \Gamma^2_{jk} \\ F^3 = mv^j \dfrac{\partial v^3}{\partial x^j} + mv^k v^j \Gamma^3_{jk} \end{cases} \qquad 13.8$$

Note that in a Cartesian system the last term in Equation 13.7, associated with the Christoffel symbol, is zero and we recover the familiar form of Newton's equation. But when we write this equation in a curvilinear coordinate system, the term associated with the Christoffel symbol is not necessarily equal to zero. This extra term acts like an inertia force and affects the path of moving objects. For example, in a spherical coordinate system, like Earth's geometry, we can write Equation 13.7 as $F^i - mv^k v^j \Gamma^i_{jk} = mv^j \dfrac{\partial v^i}{\partial x^j}$. The left-hand-side can be treated as a new total force, or $\tilde{F}^i = F^i - mv^k v^j \Gamma^i_{jk} = mv^j \dfrac{\partial v^i}{\partial x^j}$. The term $mv^k v^j \Gamma^i_{jk}$ is the *Coriolis force*, F^i_c

$$F^i_c = mv^k v^j \Gamma^i_{jk} \qquad 13.9$$

For example, from the point of view of an observer at the north pole of the Earth an object with an initial velocity along a meridian towards the equator will shift to the right, due to the inertia force resulting from the Coriolis force, while to an observer at the South Pole it will shift to the left. Or we can say that a force equivalent to the negative of the Coriolis force is acting on the object.

13.2 THE ACCELERATION VECTOR—PHYSICAL COMPONENTS

Physical components of acceleration vector can be obtained using Equation 6.2, or $a(i) = h_i a^i$. Substituting for a^i, Equation 13.5 yields $a(i) = h_i \underbrace{v^j \dfrac{\partial v^i}{\partial x^j}}_{=\frac{dv^i}{dt}} + h_i v^j v^k \Gamma^i_{jk} = h_i \dfrac{dv^i}{dt} + h_i v^j v^k \Gamma^i_{jk}$. In the latter expression, we write the velocity components in terms of their physical components as well to receive $a(i)=h_i\dfrac{d\left(v(i)/h_i\right)}{dt}+h_i\dfrac{v(j)v(k)}{h_jh_k}\Gamma^i_{jk}$. But the first term can be expanded to get $h_i\dfrac{d\left(v(i)/h_i\right)}{dt}=\dfrac{dv(i)}{dt}+h_iv(i)\dfrac{d(1/h_i)}{dt}$. Now, we define $\dfrac{dv(i)}{dt}=\dot{v}(i)$ and use Equation 13.3 to write $h_iv(i)\dfrac{d(1/h_i)}{dt}=-\dfrac{1}{h_i}v(i)\dfrac{d(h_i)}{dt}=-\dfrac{v(i)v(j)}{h_ih_j}\dfrac{\partial(h_i)}{\partial x^j}$. Now substituting into the relation for $a(i)$, we obtain

$$a(i)=\dot{v}(i)-\frac{v(i)v(j)}{h_ih_j}\frac{\partial h_i}{\partial x^j}+\frac{h_iv(j)v(k)}{h_jh_k}\Gamma^i_{jk} \qquad 13.10$$

Equation 13.10 gives the physical components of the acceleration vector in curvilinear coordinate systems. Note that for Equation 13.10 summation applies only on indices j and k, but not i.

13.3 THE ACCELERATION VECTOR IN ORTHOGONAL SYSTEMS—PHYSICAL COMPONENTS

A useful application of Equation 13.10 is for orthogonal coordinate systems. As mentioned previously, for an orthogonal system we should have $g_{ij}=g^{ij}=0$ for $i\neq j$ and we can use relations given by Equations 10.23–10.25 for Christoffel symbols. Therefore, expanding Equation 13.10, for example for $i=1$, we receive $a(1)=\dot{v}(1)-\dfrac{v(1)v(j)}{h_1h_j}\dfrac{\partial h_1}{\partial x^j}+\dfrac{h_1v(j)v(k)}{h_jh_k}\Gamma^1_{jk}$, which after writing the summations on j and k yields $a(1)=\dot{v}(1)-\dfrac{v(1)}{h_1}\underbrace{\left[\dfrac{v(1)}{h_1}\dfrac{\partial h_1}{\partial x^1}+\dfrac{v(2)}{h_2}\dfrac{\partial h_1}{\partial x^2}+\dfrac{v(3)}{h_3}\dfrac{\partial h_1}{\partial x^3}\right]}_{sum\ on\ j}$

$$+\frac{h_1v(1)}{h_1}\left[\underbrace{\frac{v(1)}{h_1}\Gamma^1_{11}+\frac{v(2)}{h_2}\Gamma^1_{12}+\frac{v(3)}{h_3}\Gamma^1_{13}}_{sum\ on\ k,\ \ j=1}\right]+\frac{h_1v(2)}{h_2}\left[\underbrace{\frac{v(1)}{h_1}\Gamma^1_{21}+\frac{v(2)}{h_2}\Gamma^1_{22}+\frac{v(3)}{h_3}\Gamma^1_{23}}_{sum\ on\ k,\ \ j=2}\right]+$$

$\dfrac{h_1 v(3)}{h_3}\left[\underbrace{\dfrac{v(1)}{h_1}\Gamma^1_{31}+\dfrac{v(2)}{h_2}\Gamma^1_{32}+\dfrac{v(3)}{h_3}\Gamma^1_{33}}_{sum\ on\ k,\ \ j=3}\right]$. But from Equations 10.23-10.25, we receive $\Gamma^1_{11}=\dfrac{1}{h_1}\dfrac{\partial h_1}{\partial x^1}, \Gamma^1_{12}=\Gamma^1_{21}=\dfrac{1}{h_1}\dfrac{\partial h_1}{\partial x^2}, \Gamma^1_{13}=\Gamma^1_{31}=\dfrac{1}{h_1}\dfrac{\partial h_1}{\partial x^3}, \Gamma^1_{22}=-\dfrac{h_2}{(h_1)^2}\dfrac{\partial h_2}{\partial x^1},$ and $\Gamma^1_{33}=-\dfrac{h_3}{(h_1)^2}\dfrac{\partial h_3}{\partial x^1}$ and $\Gamma^1_{23}=\Gamma^1_{32}=0$. Substituting for the corresponding Christoffel symbols and simplifying, we receive $a(1)=\dot{v}(1)+\dfrac{v(1)v(2)}{h_1h_2}\dfrac{\partial h_1}{\partial x^2}+\dfrac{v(1)v(3)}{h_1h_3}\dfrac{\partial h_1}{\partial x^3}-\dfrac{v(2)v(2)}{h_1h_2}\dfrac{\partial h_2}{\partial x^1}-\dfrac{v(3)v(3)}{h_1h_3}\dfrac{\partial h_3}{\partial x^1}$. By similar operation we can calculate the second and third components, i.e., $a(2)$ and $a(3)$. Equation 13.11 lists all the physical components of acceleration vector in a 3D orthogonal coordinate system, as

$$\begin{cases} a(1)=\dot{v}(1)+\dfrac{v(1)v(2)}{h_1h_2}\dfrac{\partial h_1}{\partial x^2}+\dfrac{v(1)v(3)}{h_1h_3}\dfrac{\partial h_1}{\partial x^3}-\dfrac{v(2)v(2)}{h_1h_2}\dfrac{\partial h_2}{\partial x^1}-\dfrac{v(3)v(3)}{h_1h_3}\dfrac{\partial h_3}{\partial x^1} \\ a(2)=\dot{v}(2)+\dfrac{v(1)v(2)}{h_1h_2}\dfrac{\partial h_2}{\partial x^1}+\dfrac{v(2)v(3)}{h_2h_3}\dfrac{\partial h_2}{\partial x^3}-\dfrac{v(1)v(1)}{h_1h_2}\dfrac{\partial h_1}{\partial x^2}-\dfrac{v(3)v(3)}{h_2h_3}\dfrac{\partial h_3}{\partial x^2} \\ a(3)=\dot{v}(3)+\dfrac{v(1)v(3)}{h_1h_3}\dfrac{\partial h_3}{\partial x^1}+\dfrac{v(2)v(3)}{h_2h_3}\dfrac{\partial h_3}{\partial x^2}-\dfrac{v(1)v(1)}{h_1h_3}\dfrac{\partial h_1}{\partial x^3}-\dfrac{v(2)v(2)}{h_2h_3}\dfrac{\partial h_2}{\partial x^3} \end{cases} \qquad 13.11$$

Relations given by Equation 13.11 can be written in index form as, summation on index j only,

$$a(i)=\dot{v}(i)+\frac{v(i)v(j)}{h_ih_j}\frac{\partial h_i}{\partial x^j}-\frac{v(j)v(j)}{h_ih_j}\frac{\partial h_j}{\partial x^i} \qquad 13.12$$

In the next section, we use Equation 13.12 to calculate an acceleration vector's physical components in cylindrical and spherical coordinate systems. For a Cartesian system, Equation 13.12 recovers the familiar relation (i.e., $a(i)=\dfrac{d(v(i))}{dt}$).

13.3.1 Example: Acceleration vector physical components in cylindrical and spherical coordinate systems

From the results obtained in Examples 8.1 and 8.2, we have the scale factors for cylindrical and spherical coordinates as $\left(h_r, h_\theta, h_z\right) = (1, r, 1)$ and $\left(h_r, h_\varphi, h_\theta\right) = (1, r, r\sin\varphi)$, respectively. Recall that the cylindrical coordinates are $\left(x^1, x^2, x^3\right) \equiv (r, \theta, z)$ and those of a spherical system are $\left(x^1, x^2, x^3\right) \equiv (r, \varphi, \theta)$. Now, using Equation 13.11 we can write the physical components of the acceleration vector for cylindrical coordinates as $a(1) = \dot{v}(1) - \dfrac{v(2)v(2)}{r}$, $a(2) = \dot{v}(2) + \dfrac{v(1)v(2)}{r}$, and $a(3) = \dot{v}(3)$. In terms of coordinate notation, we can write it as $a_r = \dot{v}_r - \dfrac{v_\theta^2}{r}$, $a_\theta = \dot{v}_\theta + \dfrac{v_r v_\theta}{r}$, and $a_z = \dot{v}_z$. Note that all components are the physical components of the corresponding vectors. Therefore, the acceleration vector is written as

$$\vec{a} = \left(\dot{v}_r - \frac{v_\theta^2}{r}\right)\vec{e}(r) + \left(\dot{v}_\theta + \frac{v_r v_\theta}{r}\right)\vec{e}(\theta) + \dot{v}_z \vec{e}(z)$$

Note that $\vec{e}(i)$ are unit vectors in cylindrical coordinates. We can also write the acceleration vector in terms of Cartesian unit vectors $\vec{E}_i$, by substituting for $\vec{e}(r) = \cos\theta\vec{E}_1 + \sin\theta\,\vec{E}_2$, $\vec{e}(\theta) = -\sin\theta\vec{E}_1 + \cos\theta\vec{E}_2$, and $\vec{e}(z) = \vec{E}_3$ (see Example 8.1), or

$$\begin{aligned}\vec{a} &= \left[\left(\dot{v}_r - \frac{v_\theta^2}{r}\right)\cos\theta - \left(\dot{v}_\theta + \frac{v_r v_\theta}{r}\right)\sin\theta\right]\vec{E}_1 \\ &+ \left[\left(\dot{v}_r - \frac{v_\theta^2}{r}\right)\sin\theta + \left(\dot{v}_\theta + \frac{v_r v_\theta}{r}\right)\cos\theta\right]\vec{E}_2 + \dot{v}_z\vec{E}_3\end{aligned} \qquad 13.13$$

In terms of coordinates themselves, we have

$$\begin{aligned}\vec{a} &= \left[\left(\ddot{r} - r\dot{\theta}^2\right)\cos\theta - \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\sin\theta\right]\vec{E}_1 \\ &+ \left[\left(\ddot{r} - r\dot{\theta}^2\right)\sin\theta + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\cos\theta\right]\vec{E}_2 + \ddot{z}\vec{E}_3\end{aligned}$$

Similarly, for a spherical coordinate system, using Equation 13.11, we can write the physical components of the acceleration vector for spherical coordinates as $a(1) = a_r = \dot{v}_r - \dfrac{v_\varphi^2}{r} - \dfrac{v_\theta^2}{r}$, $a(2) = a_\varphi = \dot{v}_\varphi + \dfrac{v_r v_\varphi}{r} - \dfrac{v_\theta^2\cos\varphi}{r\sin\varphi}$, and

$a(3) = a_\theta = \dot{v}_\theta + \frac{v_r v_\theta}{r} + \frac{v_\theta v_\varphi \cos\varphi}{r\sin\varphi}$. Note that all components are the physical components of the corresponding vectors. Therefore, the acceleration vector is

$$\vec{a} = \left(\dot{v}_r - \frac{v_\varphi^2 + v_\theta^2}{r}\right)\vec{e}(r) + \left(\dot{v}_\varphi + \frac{v_r v_\varphi}{r} - \frac{v_\theta^2 \cos\varphi}{r\sin\varphi}\right)\vec{e}(\varphi) + \left(\dot{v}_\theta + \frac{v_r v_\theta}{r} + \frac{v_\theta v_\varphi \cos\varphi}{r\sin\varphi}\right)\vec{e}(\theta)$$

Note that $\vec{e}(i)$ are unit vectors in spherical coordinates. In terms of the coordinates themselves, we have

$$\vec{a} = \left[\ddot{r} - r\dot{\varphi}^2 - r\sin^2\varphi\dot{\theta}^2\right]\vec{e}(r) + \left[r\ddot{\varphi} + 2\dot{r}\dot{\varphi} - r\sin\varphi\cos\varphi\dot{\theta}^2\right]\vec{e}(\varphi)$$
$$+ \left[r\ddot{\theta}\sin\varphi + 2\dot{r}\dot{\theta}\sin\varphi + 2r\dot{\varphi}\dot{\theta}\cos\varphi\right]\vec{e}(\theta).$$

We can also write the acceleration vector in terms of Cartesian unit vectors $\vec{E}_i$, by substituting for $\vec{e}(r) = \sin\varphi\cos\theta\vec{E}_1 + \sin\varphi\sin\theta\vec{E}_2 + \cos\varphi\vec{E}_3$, $\vec{e}(\varphi) = \cos\varphi\cos\theta\vec{E}_1 + \cos\varphi\sin\theta\vec{E}_2 - \sin\varphi\vec{E}_3$, and $\vec{e}(\theta) = -\sin\theta\vec{E}_1 + \cos\theta\vec{E}_2$ (see Example 8.1), or

$$\begin{aligned}\vec{a} = &\left[\left(\dot{v}_r - \frac{v_\varphi^2}{r} - \frac{v_\theta^2}{r}\right)\sin\varphi\cos\theta + \left(\dot{v}_\varphi + \frac{v_r v_\varphi}{r} - \frac{v_\theta^2\cos\varphi}{r\sin\varphi}\right)\cos\varphi\cos\theta - \left(\dot{v}_\theta + \frac{v_r v_\theta}{r} + \frac{v_\theta v_\varphi\cos\varphi}{r\sin\varphi}\right)\sin\theta\right]\vec{E}_1\\ &+\left[\left(\dot{v}_r - \frac{v_\varphi^2}{r} - \frac{v_\theta^2}{r}\right)\sin\varphi\sin\theta + \left(\dot{v}_\varphi + \frac{v_r v_\varphi}{r} - \frac{v_\theta^2\cos\varphi}{r\sin\varphi}\right)\cos\varphi\sin\theta + \left(\dot{v}_\theta + \frac{v_r v_\theta}{r} + \frac{v_\theta v_\varphi\cos\varphi}{r\sin\varphi}\right)\cos\theta\right]\vec{E}_2\\ &+(\cos\varphi - \sin\varphi)\vec{E}_3\end{aligned} \quad 13.14$$

Equations 13.13 and 13.14 are useful relations for calculating acceleration vectors in cylindrical and spherical coordinate systems, respectively. Similar calculations can be performed for arbitrary orthogonal curvilinear systems.

13.4 SUBSTANTIAL TIME DERIVATIVES OF TENSORS

Another form of derivative of tensors appearing in governing equations is the *substantial derivative*—also referred to as the *total* or *convective time derivative*. For a quantity like A—a scalar, vector, or tensor—that varies with space and time, we would like to collect all its derivatives. Let's assume that we have a body of fluid moving in a fixed coordinate system, for example, a Cartesian system. We consider two scenarios: a) we "ride" the fluid and move with it and register its variation with respect to time, and b) we stay at a fixed location in space and register the variation of the fluid passing by with respect to time. In case (a) the changes are due to any variation of the fluid w.r.t time, since we move with it and relative space change is absent/null. In other words, the change w.r.t space is implicitly included. But in case (b) we may have changes w.r.t both time and space for the fluid in motion.

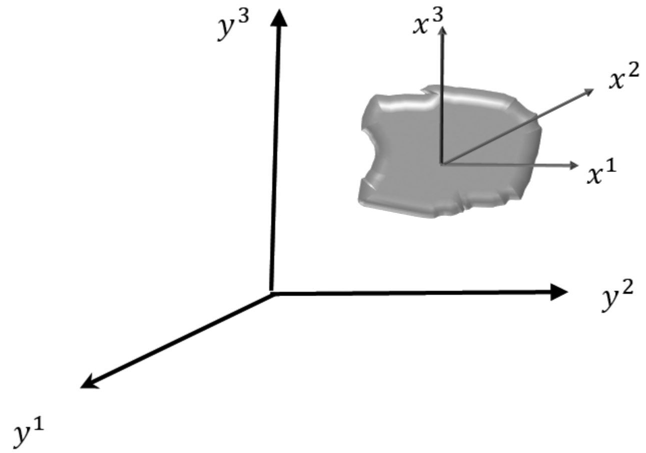


FIGURE 13.1 Local coordinates attached to a moving fluid body in a fixed Cartesian coordinate system.

Let's consider function $A = A\left(y^i, t\right)$, where A is the tensor quantity, y^i is the fixed Cartesian coordinate system, and t is time (see Figure 13.1). Here, we assume that the ratio $\dfrac{\delta y^i}{\delta t} = \upsilon^i$ applies, or the change of space coordinates of the fluid w.r.t time is the same as the local fluid velocity. Now the total time derivative is given by the following limit, $\dfrac{DA}{Dt} = \lim\limits_{\delta y^i, \delta t \to 0}\left(\dfrac{\partial A}{\partial t} + \dfrac{\partial A}{\partial y^i}\dfrac{\partial y^i}{\partial t}\right)$, or

$$\frac{DA}{Dt} = \frac{\partial A}{\partial t} + \upsilon^i \frac{\partial A}{\partial y^i} \qquad 13.15$$

An immediate extension is to write the total derivative for $A = A^i_j \vec{e}_i \vec{e}^{\,j}$, a mixed tensor of the second rank, in an arbitrary system x^i, or

$$\frac{DA}{Dt} = \frac{\partial\left(A^i_j \vec{e}_i \vec{e}^{\,j}\right)}{\partial t} + \upsilon^k \frac{\partial\left(A^i_j \vec{e}_i \vec{e}^{\,j}\right)}{\partial x^k} \qquad 13.16$$

Using Equation 10.14, we can write the second term on the right-hand-side as $\upsilon^k \dfrac{\partial\left(A^i_j \vec{e}_i \vec{e}^{\,j}\right)}{\partial x^k} = \upsilon^k A^i_{j,k} \vec{e}_i \vec{e}^{\,j}$. The first term is just equal to $\dfrac{\partial A^i_j}{\partial t}$, if the coordinates x^i are also fixed. Note that if the x^i system is moving, we should consider the changes of the basis vectors with reference to time as well (see Equation 13.16). Therefore, we get the components (mixed contravariant/covariant) of the substantial time derivative of A, as

$$\left(\frac{DA}{Dt}\right)^i_j = \frac{\partial\left(A^i_j\right)}{\partial t} + \upsilon^k A^i_{j,k} \qquad 13.17$$

Note that $A^i_{j,k} = \dfrac{\partial A^i_j}{\partial x^k} + \Gamma^i_{km} A^m_j - \Gamma^n_{kj} A^i_n$.

A specific case is to let A be equal to velocity itself, which results in the total derivative of υ^i, as

$$\left(\frac{D\upsilon}{Dt}\right)^i = \frac{\partial \upsilon^i}{\partial t} + \upsilon^j \upsilon^i_{,j} \qquad 13.18$$

In orthogonal coordinate systems, we can write the physical components the convective time derivative of a vector as

$$\begin{aligned}\left(\frac{D\vec{A}}{Dt}\right)(i) = \frac{\partial A(i)}{\partial t} + \frac{\upsilon(j)}{h_j}\frac{\partial A(i)}{\partial x^j} + \frac{\upsilon(i)A(j)}{h_i h_j}\frac{\partial h_i}{\partial x^j} \\ - \frac{\upsilon(j)A(j)}{h_i h_j}\frac{\partial h_j}{\partial x^i},\ no\,sum\,on\,i\end{aligned} \qquad 13.19$$

Using Equation 13.19 we can write the total time derivative of vectors like velocity or acceleration. The following example demonstrates these calculations.

13.4.1 Example: Substantial time derivation of acceleration vectors—physical components

From previous examples, we have $(h_r, h_\theta, h_z) = (1, r, 1)$ for cylindrical coordinate systems and $(h_r, h_\varphi, h_\theta) = (1, r, r\sin\varphi)$ for spherical coordinate systems. We can write the physical components of the total derivative of acceleration vector $\vec{a} = (a_r, a_\theta, a_z)$, in cylindrical coordinates, by using Equation 13.19, as

$$\begin{cases} \left(\dfrac{D\vec{a}}{Dt}\right)_r = \dfrac{\partial a_r}{\partial t} + \upsilon_r \dfrac{\partial a_r}{\partial r} + \dfrac{\upsilon_\theta}{r}\dfrac{\partial a_r}{\partial \theta} + \upsilon_z \dfrac{\partial a_r}{\partial z} - \upsilon_\theta \dfrac{a_\theta}{r} \\ \left(\dfrac{D\vec{a}}{Dt}\right)_\theta = \dfrac{\partial a_\theta}{\partial t} + \upsilon_r \dfrac{\partial a_\theta}{\partial r} + \dfrac{\upsilon_\theta}{r}\dfrac{\partial a_\theta}{\partial \theta} + \upsilon_z \dfrac{\partial a_\theta}{\partial z} + \dfrac{\upsilon_\theta a_r}{r} \\ \left(\dfrac{D\vec{a}}{Dt}\right)_z = \dfrac{\partial a_z}{\partial t} + \upsilon_r \dfrac{\partial a_z}{\partial r} + \dfrac{\upsilon_\theta}{r}\dfrac{\partial a_z}{\partial \theta} + \upsilon_z \dfrac{\partial a_z}{\partial z} \end{cases} \qquad 13.20$$

Similarly, for spherical coordinates we receive the physical components of the total derivative of acceleration vector as

$$\begin{cases} \left(\dfrac{D\vec{a}}{Dt}\right)_r = \dfrac{\partial a_r}{\partial t} + \upsilon_r \dfrac{\partial a_r}{\partial r} + \dfrac{\upsilon_\varphi}{r}\dfrac{\partial a_r}{\partial \varphi} + \dfrac{\upsilon_\theta}{r\sin\varphi}\dfrac{\partial a_r}{\partial \theta} - \upsilon_\varphi \dfrac{a_\varphi}{r} - \upsilon_\theta \dfrac{a_\theta}{r} \\ \left(\dfrac{D\vec{a}}{Dt}\right)_\varphi = \dfrac{\partial a_\varphi}{\partial t} + \upsilon_r \dfrac{\partial a_\varphi}{\partial r} + \dfrac{\upsilon_\varphi}{r}\dfrac{\partial a_\varphi}{\partial \varphi} + \dfrac{\upsilon_\theta}{r\sin\varphi}\dfrac{\partial a_\varphi}{\partial \theta} + \dfrac{\upsilon_\varphi a_r}{r} - \dfrac{\upsilon_\theta a_\theta \cos\varphi}{r\sin\varphi} \\ \left(\dfrac{D\vec{a}}{Dt}\right)_\theta = \dfrac{\partial a_\theta}{\partial t} + \upsilon_r \dfrac{\partial a_\theta}{\partial r} + \dfrac{\upsilon_\varphi}{r}\dfrac{\partial a_\theta}{\partial \varphi} + \dfrac{\upsilon_\theta}{r\sin\varphi}\dfrac{\partial a_\theta}{\partial \theta} + \dfrac{\upsilon_\theta a_r}{r} + \dfrac{\upsilon_\theta a_\varphi \cos\varphi}{r\sin\varphi} \end{cases} \qquad 13.21$$

Note that in these relations (i.e., Equations 13.20 and 13.21) all components are the physical components of the corresponding vectors.

13.5 CONSERVATION EQUATIONS—COORDINATE INDEPENDENT FORMS

A fundamental equation for the transformation of a tensor quantity per unit volume, Ψ in an arbitrary control volume with a velocity field $\vec{V}$, can be obtained using the Gauss divergence theorem and the law of the principle of conservation [11]. An integral equation defining the time rate and divergence of convective transformation balanced with the rate of production/destruction Ω of the quantity leads to the differential relation given by Equation 13.22, as

$$\frac{\partial \Psi}{\partial t} + \left(\Psi V^i\right)_{,i} = \Omega \qquad 13.22$$

The second term can be written as $\left(\Psi V^i\right)_{,i} = V^i \Psi_{,i} + \Psi V^i_{,i}$ since covariant differentiation follows the differentiation product rule. Therefore, L.H.S of Equation 13.22 can be written as $\underbrace{\frac{\partial \Psi}{\partial t} + V^i \Psi_{,i}}_{=\frac{D\Psi}{Dt}} + \Psi V^i{}_{,i} = \frac{D\Psi}{Dt} + \Psi V^i{}_{,i}$. In vector form, this equation reads

$$\frac{D\Psi}{Dt} + \Psi \vec{\nabla} \cdot \vec{V} = \frac{D\Psi}{Dt} + \Psi V^i{}_{,i} = \Omega \qquad 13.23$$

Equation 13.22 (or equally Equation 13.23) can be considered the equation of motion for the control volume or, in general, the medium that the quantity Ψ transports in it. The source term is associated with the quantity in question: for example, the mass, momentum, energy, electric charge, etc.

Now we consider mass conservation, or let $\Psi = \rho$ where ρ is mass per unit volume or density of the quantity in motion. After substitution in Equation 13.23, we get $\frac{D\rho}{Dt} + \rho V^i{}_{,i} = \Omega$. If mass source is zero, then we get the continuity equation, as

$$\frac{D\rho}{Dt} + \rho V^i{}_{,i} = 0 \qquad 13.24$$

Note that $\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + V^i \rho_{,i}$. Some specific cases can be observed:

- $\rho = \rho(t)$ only, i.e., we have an inhomogeneous quantity whose density changes with time. In this case the continuity equation reads as $\frac{\partial \rho}{\partial t} + V^i \underbrace{(\rho_{,i})}_{=0} + \rho V^i{}_{,i} = 0$, or $\frac{\partial \rho}{\partial t} + \rho V^i{}_{,i} = 0$.
- $\rho = \rho(x^i)$ only, i.e., we have an inhomogeneous quantity whose density changes from location to location in the continuum but remains constant with time at any given location. In this case the continuity equation reads as $\underbrace{\left(\frac{\partial \rho}{\partial t}\right)}_{=0} + V^i \rho_{,i} + \rho V^i{}_{,i} = 0$, or $\left(\rho V^i\right) = 0$.
- ρ is a constant, i.e., we have a homogeneous quantity with constant density at all locations and times in the continuum. In this case the continuity equation reads as $\underbrace{\left(\frac{D\rho}{Dt}\right)}_{=0} + \rho V^i{}_{,i} = 0$, or $V^i{}_{,i} = 0$.

Now we consider momentum conservation, or let $\Psi = \rho\vec{V}$, i.e., momentum per unit volume of the quantity in motion. In this case Ψ is a vector quantity. Substituting in Equation 13.22, the i^{th} contravariant component reads

$$\frac{\partial\left(\rho V^i\right)}{\partial t}+\left(\rho V^i V^j\right)_{,j}=\Omega^i \qquad 13.25$$

Equation 13.25 is Newton's 2nd law written in the coordinate independent form for a medium, for example a fluid moving in the continuum. The L.H.S is the inertia force and the R.H.S is the applied force on the material, per unit volume. Considering the material as a fluid, then the applied force could be, in general, a combination of hydrostatic, viscous, gravitational, electromagnetic, etc. forces.

CHAPTER 14

COLLECTION OF RELATIONS FOR SELECTED COORDINATE SYSTEMS

In this section, we provide a list of some commonly used coordinate systems as well as relations for an arbitrary orthogonal curvilinear coordinate system. We categorize the content based on the coordinate systems. All coordinates are considered in a 3D Euclidean space. Cartesian coordinate systems are fixed references consisting of three flat planes. The curvilinear coordinates may include one or more curved coordinate surfaces.

14.1 CARTESIAN COORDINATE SYSTEM

We use $\left(\vec{e}(1),\vec{e}(2),\vec{e}(3)\right)=\left(\vec{E}_x,\vec{E}_y,\vec{E}_z\right)$, $\left(A(1),A(2),A(3)\right)=\left(A_x,A_y,A_z\right)$ symbols for physical components. Note that in a Cartesian system, all contravariant, covariant, and physical components of a tensor quantity are identical, hence all indices are shown as subscripts.

TABLE 14.1 Relations for tensors and their related derivatives in Cartesian coordinate systems.

Coordinates	$(x, y, z) \equiv (X, Y, Z)$
Basis/Unit vectors, $\vec{E}_i$	$(\vec{E}_x, \vec{E}_y, \vec{E}_z) \equiv (\vec{i}, \vec{j}, \vec{k})$, orthogonal
Scale factors, h_i	$(h_x, h_y, h_z) = (1,1,1)$
Metric tensors, g_{ij}	$g_{xx} = g_{yy} = g_{zz} = 1$, the rest are null
Jacobian	$\mathcal{J} = 1$
Unit volume	$d\mathbb{V} = dxdydz$
Line element and magnitude	$\overrightarrow{dS} = dx\vec{E}_x + dy\vec{E}_y + dz\vec{E}_z$	$\lvert\overrightarrow{dS}\rvert = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$
Vector components	$\vec{A} = (A_x, A_y, A_z) = A_x\vec{E}_x + A_y\vec{E}_y + A_z\vec{E}_z$
Christoffel symbols	null
Dot-product (two vectors)	$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$
Cross-product (two vectors)	$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\vec{E}_x + (A_z B_x - A_x B_z)\vec{E}_y + (A_x B_y - A_y B_x)\vec{E}_z$
Gradient vector	$\vec{\nabla} = \vec{E}_x \frac{\partial}{\partial x} + \vec{E}_y \frac{\partial}{\partial y} + \vec{E}_z \frac{\partial}{\partial z}$
Gradient of a scalar, Ψ	$\vec{\nabla}\Psi = \vec{E}_x \frac{\partial \Psi}{\partial x} + \vec{E}_y \frac{\partial \Psi}{\partial y} + \vec{E}_z \frac{\partial \Psi}{\partial z}$
Gradient of a vector, $\vec{A}$	$\vec{\nabla}\vec{A} = \begin{bmatrix} \frac{\partial A_x}{\partial x} & \frac{\partial A_y}{\partial x} & \frac{\partial A_z}{\partial x} \\ \frac{\partial A_x}{\partial y} & \frac{\partial A_y}{\partial y} & \frac{\partial A_z}{\partial y} \\ \frac{\partial A_x}{\partial z} & \frac{\partial A_y}{\partial z} & \frac{\partial A_z}{\partial z} \end{bmatrix}$
Curl of a vector, $\vec{A}$	$\vec{\nabla} \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{E}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{E}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{E}_z$
Divergence of a vector, $\vec{A}$	$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

(continued)

Laplacian of a scalar, Ψ	$\nabla^2\Psi = \vec{\nabla}\cdot\left(\vec{\nabla}\Psi\right) = \dfrac{\partial^2\Psi}{\partial x^2} + \dfrac{\partial^2\Psi}{\partial y^2} + \dfrac{\partial^2\Psi}{\partial z^2}$
Laplacian of a vector, $\vec{A}$	$\nabla^2\vec{A} =$ $\left(\dfrac{\partial^2 A_x}{\partial x^2} + \dfrac{\partial^2 A_x}{\partial y^2} + \dfrac{\partial^2 A_x}{\partial z^2}\right)\vec{E}_x + \left(\dfrac{\partial^2 A_y}{\partial x^2} + \dfrac{\partial^2 A_y}{\partial y^2} + \dfrac{\partial^2 A_y}{\partial z^2}\right)\vec{E}_y$ $+\left(\dfrac{\partial^2 A_z}{\partial x^2} + \dfrac{\partial^2 A_z}{\partial y^2} + \dfrac{\partial^2 A_z}{\partial z^2}\right)\vec{E}_z$
Biharmonic operator	$\nabla^4 = \left(\dfrac{\partial^4}{\partial x^4} + \dfrac{\partial^4}{\partial y^4} + \dfrac{\partial^4}{\partial z^4}\right) + 2\left(\dfrac{\partial^4}{\partial x^2\partial y^2} + \dfrac{\partial^4}{\partial y^2\partial z^2} + \dfrac{\partial^4}{\partial x^2\partial z^2}\right)$

14.2 CYLINDRICAL COORDINATE SYSTEMS

We use $\left(\vec{e}(1),\vec{e}(2),\vec{e}(3)\right) = \left(\vec{e}(r),\vec{e}(\theta),\vec{e}(z)\right)$, $\vec{A} = \left(A(1),A(2),A(3)\right) = \left(A(r),A(\theta),A(z)\right)$ to designate the unit vectors $\vec{e}(i)$ and physical components of vector $\vec{A}$. Most of the expressions are written in terms of physical components. For Christoffel symbols, see Equation 10.28.

TABLE 14.2 Relations for quantities and their related derivatives in cylindrical coordinate systems.

Coordinates (orthogonal)	$\left(x^1,x^2,x^3\right) \equiv \left(r,\theta,z\right)$, θ is the azimuth angle		
Coordinate functions	$x = r\cos\theta$	$y = r\sin\theta$	$z = z$
Coordinate surfaces	$x^2 + y^2 = r^2$, cylinders	$y/x = \tan\theta$, planes	$z = constant$, planes
Basis vectors, $\vec{e}_i$, covariant	$\left(\vec{e}_r,\vec{e}_\theta,\vec{e}_z\right) = \left(\begin{Bmatrix}\cos\theta\\ \sin\theta\\ 0\end{Bmatrix},\begin{Bmatrix}-r\sin\theta\\ r\cos\theta\\ 0\end{Bmatrix},\begin{Bmatrix}0\\ 0\\ 1\end{Bmatrix}\right)$		
Basis vectors, $\vec{e}^i$, contravariant	$\left(\vec{e}^r,\vec{e}^\theta,\vec{e}^z\right) = \left(\begin{Bmatrix}\cos\theta\\ \sin\theta\\ 0\end{Bmatrix},\begin{Bmatrix}-\sin\theta/r\\ \cos\theta/r\\ 0\end{Bmatrix},\begin{Bmatrix}0\\ 0\\ 1\end{Bmatrix}\right)$		
Scale factors, h_i	$\left(h_r,h_\theta,h_z\right) = (1,r,1)$		

(continued)

Unit vectors, physical components	$\left(\vec{e}(r),\vec{e}(\theta),\vec{e}(z)\right)=\left(\begin{Bmatrix}\cos\theta\\ \sin\theta\\ 0\end{Bmatrix},\begin{Bmatrix}-\sin\theta\\ \cos\theta\\ 0\end{Bmatrix},\begin{Bmatrix}0\\ 0\\ 1\end{Bmatrix}\right)$
Metric tensors, g_{ij} and g^{ij}	$g_{rr}=g_{zz}=\left(g^{rr}\right)^{-1}=\left(g^{zz}\right)^{-1}=1, g_{\theta\theta}=\left(g^{\theta\theta}\right)^{-1}=r^2$, the rest are null-system is orthogonal
Jacobian	$\mathcal{J}=r$
Unit volume	$d\mathbb{V}=rdrd\theta dz$
Line element and magnitude	$\overrightarrow{dS}=dr\vec{e}(r)+rd\theta\vec{e}(\theta)+dz\vec{e}(z)$	$\left\lvert\overrightarrow{dS}\right\rvert=\sqrt{(dr)^2+(rd\theta)^2+(dz)^2}$
Vector components	$\vec{A}=A(r)\vec{e}(r)+A(\theta)\vec{e}(\theta)+A(z)\vec{e}(z)$
Christoffel symbols, based on Unit vectors	$\hat{\Gamma}^r_{\theta\theta}=-1,\ \hat{\Gamma}^\theta_{r\theta}=\hat{\Gamma}^\theta_{\theta r}=1$, the rest are null
Christoffel symbols, based on covariant Basis vectors	$\Gamma^r_{\theta\theta}=-r,\ \Gamma^\theta_{r\theta}=\Gamma^\theta_{\theta r}=1/r$, the rest are null
Dot-product (two vectors)	$\vec{A}\cdot\vec{B}=A(r)B(r)+A(\theta)B(\theta)+A(z)B(z)$
Cross-product (two vectors)	$\begin{aligned}\vec{A}\times\vec{B}=&\left[A(\theta)B(z)-A(z)B(\theta)\right]\vec{e}(r)\\ &+\left[A(z)B(r)-A(r)B(z)\right]\vec{e}(\theta)\\ &+\left[A(r)B(\theta)-A(\theta)B(r)\right]\vec{e}(z)\end{aligned}$
Gradient vector	$\vec{\nabla}=\vec{e}_r\dfrac{\partial}{\partial r}+\dfrac{1}{r}\vec{e}_\theta\dfrac{\partial}{\partial\theta}+\vec{e}_z\dfrac{\partial}{\partial z}$
Gradient of a scalar, Ψ	$\vec{\nabla}\Psi=\vec{e}_r\dfrac{\partial\Psi}{\partial r}+\dfrac{1}{r}\vec{e}_\theta\dfrac{\partial\Psi}{\partial\theta}+\vec{e}_z\dfrac{\partial\Psi}{\partial z}$
Gradient of a vector, $\vec{A}$	$\vec{\nabla}\vec{A}=\begin{bmatrix}\dfrac{\partial A(r)}{\partial r} & \dfrac{\partial A(\theta)}{\partial r} & \dfrac{\partial A(z)}{\partial r}\\ \dfrac{\partial A(r)}{r\partial\theta}-\dfrac{A(\theta)}{r} & \dfrac{\partial A(\theta)}{r\partial\theta}+\dfrac{A(r)}{r} & \dfrac{\partial A(z)}{r\partial\theta}\\ \dfrac{\partial A(r)}{\partial z} & \dfrac{\partial A(\theta)}{\partial z} & \dfrac{\partial A(z)}{\partial z}\end{bmatrix}$

(continued)

Curl of a vector, $\vec{A}$	$\vec{\nabla}\times\vec{A}=\left(\frac{\partial A(z)}{r\partial\theta}-\frac{\partial A(\theta)}{\partial z}\right)\vec{e}(r)+\left(\frac{\partial A(r)}{\partial z}-\frac{\partial A(z)}{\partial r}\right)\vec{e}(\theta)+\left(\frac{\partial(rA(\theta))}{r\partial r}-\frac{\partial A(r)}{r\partial\theta}\right)\vec{e}(z)$
Divergence of a vector, $\vec{A}$	$\vec{\nabla}\cdot\vec{A}=\frac{\partial(rA(r))}{r\partial r}+\frac{\partial A(\theta)}{r\partial\theta}+\frac{\partial A(z)}{\partial z}$
Laplacian operator	$\nabla^2=\vec{\nabla}\cdot(\vec{\nabla})=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}$
Laplacian of a scalar, Ψ	$\nabla^2\Psi=\vec{\nabla}\cdot(\vec{\nabla}\Psi)=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}+\frac{\partial^2\Psi}{\partial z^2}$
Laplacian of a vector, $\vec{A}$	$\nabla^2\vec{A}=\left[\nabla^2A(r)-\frac{2}{r^2}\frac{\partial A(\theta)}{\partial\theta}-\frac{A(r)}{r^2}\right]\vec{e}(r)+\left[\nabla^2A(\theta)+\frac{2}{r^2}\frac{\partial A(r)}{\partial\theta}-\frac{A(\theta)}{r^2}\right]\vec{e}(\theta)+\left[\nabla^2A(z)\right]\vec{e}(z)$
Biharmonic operator	$\nabla^4=\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right]\right]+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left[\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]+\frac{\partial^4}{\partial z^4}$

14.3 SPHERICAL COORDINATE SYSTEMS

We use $(\vec{e}(1),\vec{e}(2),\vec{e}(3))=(\vec{e}_r,\vec{e}_\varphi,\vec{e}_\theta)$, $(A(1),A(2),A(3))=(A(r),A(\varphi),A(\theta))$ to designate the unit vectors $\vec{e}(i)$ and physical components of vector $\vec{A}$. Most of the expressions are written in terms of physical components. For Christoffel symbols, see Equation 10.28.

TABLE 14.3 Relations for quantities and their related derivatives in spherical coordinate systems.

Coordinates (orthogonal)	(r,φ,θ), θ azimuth and φ polar angle		
Coordinate functions	$x=r\sin\varphi\cos\theta$	$y=r\sin\varphi\sin\theta$	$z=r\cos\varphi$
Coordinate surfaces	$x^2+y^2+z^2=r^2$, spheres	$(x^2+y^2)/z^2=\tan^2\varphi$, cones	$y/x=\tan\theta$, planes

(continued)

Basis vectors, $\vec{e}_i$, covariant	$\left(\vec{e}_r,\vec{e}_\varphi,\vec{e}_\theta\right)=\left(\begin{Bmatrix}\sin\varphi\cos\theta\\ \sin\varphi\sin\theta\\ \cos\varphi\end{Bmatrix},\begin{Bmatrix}r\cos\varphi\cos\theta\\ r\cos\varphi\sin\theta\\ -r\sin\varphi\end{Bmatrix},\begin{Bmatrix}-r\sin\varphi\sin\theta\\ r\sin\varphi\cos\theta\\ 0\end{Bmatrix}\right)$
Basis vectors, $\vec{e}^i$, contravariant	$\left(\vec{e}^r,\vec{e}^\varphi,\vec{e}^\theta\right)=\left(\begin{Bmatrix}\sin\varphi\cos\theta\\ \sin\varphi\sin\theta\\ \cos\varphi\end{Bmatrix},\frac{1}{r}\begin{Bmatrix}\cos\varphi\cos\theta\\ \cos\varphi\sin\theta\\ -\sin\varphi\end{Bmatrix},\frac{1}{r\sin\varphi}\begin{Bmatrix}-\sin\theta\\ \cos\theta\\ 0\end{Bmatrix}\right)$
Scale factors, h_i	$\left(h_r,h_\varphi,h_\theta\right)=\left(1,r,r\sin\varphi\right)$
Unit vectors, physical components	$\left(\vec{e}(r),\vec{e}(\varphi),\vec{e}(\theta)\right)=\left(\begin{Bmatrix}\sin\varphi\cos\theta\\ \sin\varphi\sin\theta\\ \cos\varphi\end{Bmatrix},\begin{Bmatrix}\cos\varphi\cos\theta\\ \cos\varphi\sin\theta\\ -\sin\varphi\end{Bmatrix},\begin{Bmatrix}-\sin\theta\\ \cos\theta\\ 0\end{Bmatrix}\right)$
Metric tensors, g_{ij} and g^{ij}	$g_{rr}=\left(g^{rr}\right)^{-1}=1, g_{\varphi\varphi}=\left(g^{\varphi\varphi}\right)^{-1}=r^2, g_{\theta\theta}=\left(g^{\theta\theta}\right)^{-1}=\left(r\sin\varphi\right)^2$, the rest are null-system is orthogonal
Jacobian	$\mathcal{J}=r^2\sin\varphi$
Unit volume	$d\mathbb{V}=r^2\sin\varphi\, dr\, d\varphi\, d\theta$
Line element and magnitude	$\overrightarrow{dS}=dr\vec{e}(r)+rd\varphi\vec{e}(\varphi)+r\sin\varphi d\theta\vec{e}(\theta)$ $\left\lvert\overrightarrow{dS}\right\rvert=\sqrt{(dr)^2+(rd\varphi)^2+(r\sin\varphi d\theta)^2}$
Vector components	$\vec{A}=A(r)\vec{e}(r)+A(\varphi)\vec{e}(\varphi)+A(\theta)\vec{e}(\theta)$
Christoffel symbols, based on covariant Basis vectors	$\Gamma^r_{\varphi\varphi}=-r$; $\Gamma^r_{\theta\theta}=-r\sin^2\varphi$; $\Gamma^\varphi_{\theta\theta}=-\sin\varphi\cos\varphi$; $\Gamma^\varphi_{r\varphi}=\Gamma^\theta_{r\theta}=1/r$; $\Gamma^\theta_{\varphi\theta}=\cot\varphi$
Christoffel symbols, based on Unit vectors	$\Gamma^r_{\varphi\varphi}=-1$; $\Gamma^r_{\theta\theta}=-\sin\varphi$; $\Gamma^\varphi_{\theta\theta}=-\cos\varphi$; $\Gamma^\varphi_{r\varphi}=1$, $\Gamma^\theta_{r\theta}=\sin\varphi$; $\Gamma^\theta_{\varphi\theta}=\cos\varphi$
Dot-product (two vectors)	$\vec{A}\cdot\vec{B}=A(r)B(r)+A(\varphi)B(\varphi)+A(\theta)B(\theta)$
Cross-product (two vectors)	$\vec{A}\times\vec{B}=\left[A(\varphi)B(\theta)-A(\theta)B(\varphi)\right]\vec{e}(r)+\left[A(\theta)B(r)-A(r)B(\theta)\right]\vec{e}(\varphi)+\left[A(r)B(\varphi)-A(\varphi)B(r)\right]\vec{e}(\theta)$

(continued)

Gradient vector	$\vec{\nabla}=\left(\frac{\partial}{\partial r}\right)\vec{e}_r+\left(\frac{1}{r}\frac{\partial}{\partial\varphi}\right)\vec{e}_\varphi+\left(\frac{1}{r\sin\varphi}\frac{\partial}{\partial\theta}\right)\vec{e}_\theta$
Gradient of a scalar, Ψ	$\vec{\nabla}\Psi=\frac{\partial\Psi}{\partial r}\vec{e}_r+\left(\frac{1}{r}\frac{\partial\Psi}{\partial\varphi}\right)\vec{e}_\varphi+\left(\frac{1}{r\sin\varphi}\frac{\partial\Psi}{\partial\theta}\right)\vec{e}_\theta$
Gradient of a vector, $\vec{A}$	$\vec{\nabla}\vec{A}=\begin{bmatrix}\frac{\partial A(r)}{\partial r} & \frac{\partial A(\varphi)}{\partial r} & \frac{\partial A(\theta)}{\partial r}\\ \frac{\partial A(r)}{r\partial\varphi}-\frac{A(\varphi)}{r} & \frac{\partial A(\varphi)}{r\partial\varphi}+\frac{A(r)}{r} & \frac{\partial A(\theta)}{r\partial\varphi}\\ \frac{1}{r\sin\varphi}\frac{\partial A(r)}{\partial\theta}-\frac{A(\theta)}{r} & \frac{1}{r\sin\varphi}\frac{\partial A(\varphi)}{\partial\theta}-\frac{A(\theta)}{r}\cot\varphi & \frac{1}{r\sin\varphi}\frac{\partial A(\theta)}{\partial\theta}+\frac{A(r)}{r}+\frac{A(\varphi)}{r}\cot\varphi\end{bmatrix}$
Curl	$\vec{\nabla}\times\vec{A}=\frac{1}{r\sin\varphi}\left(\frac{\partial\left(A(\theta)\sin\varphi\right)}{\partial\varphi}-\frac{\partial A(\varphi)}{\partial\theta}\right)\vec{e}(r)+\frac{1}{r}\left(\frac{1}{\sin\varphi}\frac{\partial A(r)}{\partial\theta}-\frac{\partial\left(rA(\theta)\right)}{\partial r}\right)\vec{e}(\varphi)+\frac{1}{r}\left(\frac{\partial\left(rA(\varphi)\right)}{\partial r}-\frac{\partial A(r)}{\partial\varphi}\right)\vec{e}(\theta)$
Divergence of a vector, $\vec{A}$	$\vec{\nabla}\cdot\vec{A}=\frac{1}{r^2}\frac{\partial\left(r^2A(r)\right)}{\partial r}+\frac{1}{r\sin\varphi}\frac{\partial\left(A(\varphi)\sin\varphi\right)}{\partial\varphi}+\frac{1}{r\sin\varphi}\frac{\partial A(\theta)}{\partial\theta}$
Divergence of a 2nd rank tensor, $\boldsymbol{A}$	$\vec{\nabla}\cdot\boldsymbol{A}=\left(\frac{\partial\left(r^2A(rr)\right)}{r^2\partial r}+\frac{1}{r\sin\varphi}\frac{\partial\left(A(r\varphi)\sin\varphi\right)}{r\partial\varphi}-\frac{A(\varphi\varphi)+A(\theta\theta)}{r}+\frac{1}{r\sin\varphi}\frac{\partial A(r\theta)}{\partial\theta}\right)\vec{e}(r)+\left(\frac{\partial\left(r^2A(r\varphi)\right)}{r^2\partial r}+\frac{1}{r\sin\varphi}\frac{\partial\left(A(\varphi\varphi)\sin\varphi\right)}{r\partial\varphi}+\frac{A(r\varphi)}{r}+\frac{1}{r\sin\varphi}\frac{\partial A(\varphi\theta)}{\partial\theta}-\frac{A(\theta\theta)}{r}\cot\varphi\right)\vec{e}(\varphi)+\left(\frac{\partial\left(r^2A(r\theta)\right)}{r^2\partial r}+\frac{\partial A(\varphi\theta)}{r\partial\varphi}+\frac{1}{r\sin\varphi}\frac{\partial A(\theta\theta)}{\partial\theta}+\frac{A(r\theta)}{r}+\frac{2A(\varphi\theta)}{r}\cot\varphi\right)\vec{e}(\theta)$
Laplacian operator	$\nabla^2=\vec{\nabla}\cdot\left(\vec{\nabla}\right)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}\right)+\frac{1}{r^2\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}$
Laplacian of a scalar, Ψ	$\nabla^2\Psi=\vec{\nabla}\cdot\left(\vec{\nabla}\Psi\right)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right)+\frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial\Psi}{\partial\varphi}\right)+\frac{1}{r^2\sin^2\varphi}\frac{\partial^2\Psi}{\partial\theta^2}$

(continued)

Laplacian of a vector, $\vec{A}$	$\nabla^2\vec{A} = \left[\nabla^2 A(r) - \frac{2A(r)}{r^2} - \frac{2}{r^2\sin\varphi}\frac{\partial(\sin\varphi A(\varphi))}{\partial\varphi} - \frac{2}{r^2\sin\varphi}\frac{\partial A(\theta)}{\partial\theta}\right]\vec{e}(r)$ $+\left[\nabla^2 A(\varphi) - \frac{A(\varphi)}{r^2\sin^2\varphi} + + \frac{2}{r^2}\frac{\partial(A(r))}{\partial\varphi} - \frac{2\cos\varphi}{r^2\sin^2\varphi}\frac{\partial A(\theta)}{\partial\theta}\right]\vec{e}(\varphi)$ $+\left[\nabla^2 A(\theta) - \frac{A(\theta)}{r^2\sin^2\varphi} + \frac{2}{r^2\sin\varphi}\frac{\partial A(r)}{\partial\theta} + \frac{2\cos\varphi}{r^2\sin^2\varphi}\frac{\partial(A(\varphi))}{\partial\theta}\right]\vec{e}(\theta)$
Biharmonic operator, ∇^4	$\nabla^4 = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right]\right]$ $+\frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left[\sin\varphi\frac{\partial}{\partial\varphi}\left[\frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}\right)\right]\right]$ $+\frac{1}{r^2\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}\left[\frac{1}{r^2\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}\right]$

14.4 PARABOLIC COORDINATE SYSTEMS

We use $(\vec{e}(1),\vec{e}(2),\vec{e}(3)) = (\vec{e}(\xi),\vec{e}(\eta),\vec{e}(\theta))$, $(A(1),A(2),A(3)) = (A(\xi),A(\eta),A(\theta))$ to designate the unit vectors $\vec{e}(i)$ and physical components of vector $\vec{A}$. Most of the expressions are written in terms of physical components. For Christoffel symbols, see Example 15.4.

TABLE 14.4 Relations for quantities and their related derivatives in parabolic coordinate systems.

Coordinates (orthogonal)	(ξ,η,θ)		
Coordinate functions	$x = \xi\eta\cos\theta$	$y = \xi\eta\sin\theta$	$z = (\xi^2 - \eta^2)/2$
Coordinate surfaces	$x^2+y^2 = -2\xi^2(z-\xi^2/2)$, paraboloids	$x^2+y^2 = -2\eta^2(z-\eta^2/2)$, paraboloids	$y/x = \tan\theta$, planes
Basis vectors, $\vec{e}_i$-covariant	$(\vec{e}_\xi,\vec{e}_\eta,\vec{e}_\theta) = \left(\begin{Bmatrix}\eta\cos\theta\\ \eta\sin\theta\\ \xi\end{Bmatrix},\begin{Bmatrix}\xi\cos\theta\\ \xi\sin\theta\\ -\eta\end{Bmatrix},\begin{Bmatrix}-\xi\eta\sin\theta\\ \xi\eta\cos\theta\\ 0\end{Bmatrix}\right)$		

(continued)

Basis vectors, $\vec{e}^{\,i}$, contravariant	$\left(\vec{e}^{\,\xi},\vec{e}^{\,\eta},\vec{e}^{\,\theta}\right)=\left(\left(\frac{1}{\xi^2+\eta^2}\begin{Bmatrix}\eta\cos\theta\\ \eta\sin\theta\\ \xi\end{Bmatrix},\frac{1}{\xi^2+\eta^2}\begin{Bmatrix}\xi\cos\theta\\ \xi\sin\theta\\ -\eta\end{Bmatrix},\frac{1}{\xi\eta}\begin{Bmatrix}-\sin\theta\\ \cos\theta\\ 0\end{Bmatrix}\right)\right)$
Scale factors, h_i	$\left(h_\xi,h_\eta,h_\theta\right)=\left(\sqrt{\xi^2+\eta^2},\sqrt{\xi^2+\eta^2},\xi\eta\right)$
Unit vectors, physical components	$\left(\vec{e}(\xi),\vec{e}(\eta),\vec{e}(\theta)\right)=$ $\left(\frac{1}{\sqrt{\xi^2+\eta^2}}\begin{Bmatrix}\eta\cos\theta\\ \eta\sin\theta\\ \xi\end{Bmatrix},\frac{1}{\sqrt{\xi^2+\eta^2}}\begin{Bmatrix}\xi\cos\theta\\ \xi\sin\theta\\ -\eta\end{Bmatrix},\begin{Bmatrix}-\sin\theta\\ \cos\theta\\ 0\end{Bmatrix}\right)$
Metric tensors, g_{ij} and g^{ij}	$g_{\xi\xi}=\left(g^{\xi\xi}\right)^{-1}=\xi^2+\eta^2, g_{\eta\eta}=\left(g^{\eta\eta}\right)^{-1}=\xi^2+\eta^2, g_{\theta\theta}=\left(g^{\theta\theta}\right)^{-1}=(\xi\eta)^2$, the rest are null
Basis vectors, $\vec{e}^{\,i}$, contravariant	$\left(\vec{e}^{\,\xi},\vec{e}^{\,\eta},\vec{e}^{\,\theta}\right)=\left(\left(\frac{1}{\xi^2+\eta^2}\begin{Bmatrix}\eta\cos\theta\\ \eta\sin\theta\\ \xi\end{Bmatrix},\frac{1}{\xi^2+\eta^2}\begin{Bmatrix}\xi\cos\theta\\ \xi\sin\theta\\ -\eta\end{Bmatrix},\frac{1}{\xi\eta}\begin{Bmatrix}-\sin\theta\\ \cos\theta\\ 0\end{Bmatrix}\right)\right)$
Jacobian	$\mathcal{J}=\xi\eta\left(\xi^2+\eta^2\right)$
Unit volume	$d\mathbb{V}=\xi\eta\left(\xi^2+\eta^2\right)d\xi d\eta d\theta$
Line element and magnitude	$\overrightarrow{dS}=\sqrt{\xi^2+\eta^2}\left[d\xi\vec{e}(\xi)+d\eta\vec{e}(\eta)\right]+\xi\eta d\theta\vec{e}(\theta)$ $\left\|\overrightarrow{dS}\right\|=\sqrt{\left(\xi^2+\eta^2\right)(d\xi)^2+\left(\xi^2+\eta^2\right)(d\eta)^2+(\xi\eta d\theta)^2}$
Vector components	$\vec{A}=A(\xi)\vec{e}(\xi)+A(\eta)\vec{e}(\eta)+A(\theta)\vec{e}(\theta)$
Christoffel symbols, based on covariant Basis vectors	$\Gamma^\xi_{\xi\xi}=-\Gamma^\xi_{\eta\eta}=\Gamma^\eta_{\xi\eta}=\Gamma^\eta_{\eta\xi}=\xi/\left(\xi^2+\eta^2\right)$ $\Gamma^\xi_{\xi\eta}=\Gamma^\xi_{\eta\xi}=\Gamma^\eta_{\eta\eta}=\eta/\left(\xi^2+\eta^2\right)$ $\Gamma^\xi_{\theta\theta}=-\xi\eta^2/\left(\xi^2+\eta^2\right)$ $\Gamma^\eta_{\theta\theta}=-\eta\xi^2/\left(\xi^2+\eta^2\right)$ $\Gamma^\theta_{\xi\theta}=\Gamma^\theta_{\theta\xi}=1/\xi$ $\Gamma^\theta_{\eta\theta}=\Gamma^\theta_{\theta\eta}=1/\eta$ $\Gamma^\theta_{\eta\theta}=\Gamma^\theta_{\theta\eta}=1/\eta$
Dot-product (two vectors)	$\vec{A}\cdot\vec{B}=A(\xi)B(\xi)+A(\eta)B(\eta)+A(\theta)B(\theta)$

(continued)

Cross-product (two vectors)	$\vec{A}\times\vec{B}=\left[A(\eta)B(\theta)-A(\theta)B(\eta)\right]\vec{e}(\xi)+\left[A(\theta)B(\xi)-A(\xi)B(\theta)\right]\vec{e}(\eta)+\left[A(\xi)B(\eta)-A(\eta)B(\xi)\right]\vec{e}(\theta)$
Gradient vector	$\vec{\nabla}=\left(\frac{1}{\sqrt{\xi^2+\eta^2}}\frac{\partial}{\partial r}\right)\vec{e}_\xi+\left(\frac{1}{\sqrt{\xi^2+\eta^2}}\frac{\partial}{\partial\eta}\right)\vec{e}_\eta+\left(\frac{1}{\xi\eta}\frac{\partial}{\partial\varphi}\right)\vec{e}_\varphi$
Gradient of a scalar, Ψ	$\vec{\nabla}\Psi=\left(\frac{1}{\sqrt{\xi^2+\eta^2}}\frac{\partial\Psi}{\partial r}\right)\vec{e}_\xi+\left(\frac{1}{\sqrt{\xi^2+\eta^2}}\frac{\partial\Psi}{\partial\eta}\right)\vec{e}_\eta+\left(\frac{1}{\xi\eta}\frac{\partial\Psi}{\partial\varphi}\right)\vec{e}_\varphi$
Laplacian operator	$\nabla^2=\vec{\nabla}\cdot\left(\vec{\nabla}\right)=\frac{1}{\xi\left(\xi^2+\eta^2\right)}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right)+\frac{1}{\eta\left(\xi^2+\eta^2\right)}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial}{\partial\eta}\right)+\frac{1}{(\xi\eta)^2}\frac{\partial^2}{\partial\theta^2}$
Laplacian of a scalar, Ψ	$\nabla^2\Psi=\vec{\nabla}\cdot\left(\vec{\nabla}\Psi\right)=\frac{1}{\xi\left(\xi^2+\eta^2\right)}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\Psi}{\partial\xi}\right)+\frac{1}{\eta\left(\xi^2+\eta^2\right)}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\Psi}{\partial\eta}\right)+\frac{1}{(\xi\eta)^2}\frac{\partial^2\Psi}{\partial\theta^2}$
Laplacian of a vector, $\vec{A}$	$\nabla^2\vec{A}=$, use Equation 11.21.

For this coordinate system, we didn't list all relations, since they are lengthy. But readers can use the relations given in the next section, as they are applicable to all orthogonal coordinate systems.

14.5 ORTHOGONAL CURVILINEAR COORDINATE SYSTEMS

We use $(x_1,x_2,x_3)\equiv(x^1,x^2,x^3)$ symbols for presentation. Note that all components listed in Table 14.5 are given as physical components. The functional relations for coordinate variables vs. those of Cartesian ones are required for calculating the values of quantities listed in Table 14.5.

TABLE 14.5 Relations for quantities and their related derivatives in curvilinear orthogonal coordinate systems.

Coordinates (orthogonal)	$(x_1, x_2, x_3) \equiv (x^1, x^2, x^3)$
Basis vectors $\vec{e}_i$, covariant	$(\vec{e}_1, \vec{e}_2, \vec{e}_3)$
Scale factors, h_i	(h_1, h_2, h_3)
Unit vectors $\vec{e}(i)$, physical components	$(\vec{e}(1), \vec{e}(2), \vec{e}(3)) = (\vec{e}_1 / h_1, \vec{e}_2 / h_2, \vec{e}_3 / h_3)$
Metric tensors, g_{ij} and g^{ij}	$g_{11} = (g^{11})^{-1} = (h_1)^2$, $g_{22} = (g^{22})^{-1} = (h_2)^2$, $g_{33} = (g^{33})^{-1} = (h_3)^2$, the rest are null
Basis vectors $\vec{e}^i$ contravariant	$(\vec{e}^1, \vec{e}^2, \vec{e}^3) = (\vec{e}_1 g^{11}, \vec{e}_2 g^{22}, \vec{e}_3 g^{33})$
Jacobian	$\mathcal{J} = h_1 h_2 h_3$
Unit volume	$d\mathbb{V} = \mathcal{J} dx_1 dx_2 dx_3$
Line element and magnitude	$\overrightarrow{dS} = h_1 dx_1 \vec{e}(1) + h_2 dx_2 \vec{e}(2) + h_3 dx_3 \vec{e}(3)$; $\lvert\overrightarrow{dS}\rvert = \sqrt{(h_1 dx_1)^2 + (h_2 dx_2)^2 + (h_3 dx_3)^2}$
Vector components	$\vec{A} = A(1)\vec{e}(1) + A(2)\vec{e}(2) + A(3)\vec{e}(3)$
Christoffel symbols	(use Equations 10.20-10.25)
Dot-product (two vectors)	$\vec{A} \cdot \vec{B} = A(1)B(1) + A(2)B(2) + A(3)B(3)$
Cross-product (two vectors)	$\vec{A} \times \vec{B} = [A(2)B(3) - A(3)B(2)]\vec{e}(1) + [A(3)B(1) - A(1)B(3)]\vec{e}(2) + [A(1)B(2) - A(2)B(1)]\vec{e}(3)$
Gradient vector	$\vec{\nabla} = \frac{1}{h_1}\frac{\partial}{\partial x_1}\vec{e}(1) + \frac{1}{h_2}\frac{\partial}{\partial x_2}\vec{e}(2) + \frac{1}{h_3}\frac{\partial}{\partial x_3}\vec{e}(3)$
Gradient of a scalar, Ψ	$\vec{\nabla}\Psi = \frac{1}{h_1}\frac{\partial \Psi}{\partial x_1}\vec{e}(1) + \frac{1}{h_2}\frac{\partial \Psi}{\partial x_2}\vec{e}(2) + \frac{1}{h_3}\frac{\partial \Psi}{\partial x_3}\vec{e}(3)$

(continued)

Gradient of a vector, $\vec{A}$	$\vec{\nabla}\vec{A} = \left(\nabla_j \vec{A}\right)\vec{e}^{\,j} = \nabla_j \left(A^i\vec{e}_i\right)\vec{e}^{\,j} = \left(\dfrac{\partial A^i}{\partial x^j} + A^k\Gamma^i_{kj}\right)\vec{e}_i\vec{e}^{\,j}$, see Example 15.13.
Curl of a vector, $\vec{A}$	$\begin{aligned} \vec{\nabla}\times\vec{A} = & \frac{1}{h_2h_3}\left[\frac{\partial\left(h_3A(3)\right)}{\partial x_2} - \frac{\partial\left(h_2A(2)\right)}{\partial x_3}\right]\vec{e}(1) \\ & + \frac{1}{h_1h_3}\left[\frac{\partial\left(h_1A(1)\right)}{\partial x_3} - \frac{\partial\left(h_3A(3)\right)}{\partial x_1}\right]\vec{e}(2) \\ & + \frac{1}{h_1h_2}\left[\frac{\partial\left(h_2A(2)\right)}{\partial x_1} - \frac{\partial\left(h_1A(1)\right)}{\partial x_2}\right]\vec{e}(3) \end{aligned}$
Divergence of a vector, $\vec{A}$	$\vec{\nabla}\cdot\vec{A} = \dfrac{1}{h_1h_2h_3}\left[\dfrac{\partial\left(h_2h_3A(1)\right)}{\partial x_1} + \dfrac{\partial\left(h_1h_3A(2)\right)}{\partial x_2} + \dfrac{\partial\left(h_1h_2A(3)\right)}{\partial x_3}\right]$
Divergence of a 2nd rank tensor, $\boldsymbol{A}$	$\begin{aligned} \vec{\nabla}\cdot\boldsymbol{A} = & \left[\begin{aligned} & \frac{1}{h_1h_2h_3}\left(\frac{\partial\left(h_2h_3A(11)\right)}{\partial x_1} + \frac{\partial\left(h_1h_3A(21)\right)}{\partial x_2} + \frac{\partial\left(h_1h_2A(31)\right)}{\partial x_3}\right) \\ & + \frac{A(12)}{h_1h_2}\frac{\partial h_1}{\partial x_2} + \frac{A(31)}{h_1h_3}\frac{\partial h_1}{\partial x_3} - \frac{A(22)}{h_1h_2}\frac{\partial h_2}{\partial x_1} - \frac{A(33)}{h_1h_3}\frac{\partial h_3}{\partial x_1} \end{aligned}\right]\vec{e}(1) \\ & + \left[\begin{aligned} & \frac{1}{h_1h_2h_3}\left(\frac{\partial\left(h_2h_3A(12)\right)}{\partial x_1} + \frac{\partial\left(h_1h_3A(22)\right)}{\partial x_2} + \frac{\partial\left(h_1h_2A(32)\right)}{\partial x_3}\right) \\ & + \frac{A(23)}{h_2h_3}\frac{\partial h_2}{\partial x_3} + \frac{A(12)}{h_1h_2}\frac{\partial h_2}{\partial x_1} - \frac{A(33)}{h_2h_3}\frac{\partial h_3}{\partial x_2} - \frac{A(11)}{h_1h_2}\frac{\partial h_1}{\partial x_2} \end{aligned}\right]\vec{e}(2) \\ & + \left[\begin{aligned} & \frac{1}{h_1h_2h_3}\left(\frac{\partial\left(h_2h_3A(13)\right)}{\partial x_1} + \frac{\partial\left(h_1h_3A(23)\right)}{\partial x_2} + \frac{\partial\left(h_1h_2A(33)\right)}{\partial x_3}\right) \\ & + \frac{A(31)}{h_1h_3}\frac{\partial h_3}{\partial x_1} + \frac{A(23)}{h_2h_3}\frac{\partial h_3}{\partial x_2} - \frac{A(11)}{h_3h_1}\frac{\partial h_1}{\partial x_3} - \frac{A(22)}{h_2h_3}\frac{\partial h_2}{\partial x_3} \end{aligned}\right]\vec{e}(3) \end{aligned}$
Laplacian of a scalar, Ψ	$\begin{aligned} \nabla^2\Psi & = \vec{\nabla}\cdot\left(\vec{\nabla}\Psi\right) \\ & = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Psi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\Psi}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Psi}{\partial x_3}\right)\right] \end{aligned}$

(continued)

Laplacian of a vector, $\vec{A}$	$\nabla^2\vec{A} = \left\{\frac{1}{h_1}\frac{\partial}{\partial x_1}\left(\vec{\nabla}\cdot\vec{A}\right)+\frac{1}{h_2h_3}\left[\begin{array}{l}\frac{\partial}{\partial x_3}\left(\frac{h_2}{h_1h_3}\left(\frac{\partial(h_1A_1)}{\partial x_3}-\frac{\partial(h_3A_3)}{\partial x_1}\right)\right)\\-\frac{\partial}{\partial x_2}\left(\frac{h_3}{h_1h_2}\left(\frac{\partial(h_2A_2)}{\partial x_1}-\frac{\partial(h_1A_1)}{\partial x_2}\right)\right)\end{array}\right]\right\}\vec{e}_1 + \left\{\frac{1}{h_2}\frac{\partial}{\partial x_2}\left(\vec{\nabla}\cdot\vec{A}\right)+\frac{1}{h_1h_3}\left[\begin{array}{l}\frac{\partial}{\partial x_1}\left(\frac{h_3}{h_1h_2}\left(\frac{\partial(h_2A_2)}{\partial x_1}-\frac{\partial(h_1A_1)}{\partial x_2}\right)\right)\\-\frac{\partial}{\partial x_3}\left(\frac{h_1}{h_2h_3}\left(\frac{\partial(h_3A_3)}{\partial x_2}-\frac{\partial(h_2A_2)}{\partial x_3}\right)\right)\end{array}\right]\right\}\vec{e}_2 + \left\{\frac{1}{h_3}\frac{\partial}{\partial x_3}\left(\vec{\nabla}\cdot\vec{A}\right)+\frac{1}{h_2h_1}\left[\begin{array}{l}\frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2h_3}\left(\frac{\partial(h_3A_3)}{\partial x_2}-\frac{\partial(h_2A_2)}{\partial x_3}\right)\right)\\-\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1h_3}\left(\frac{\partial(h_1A_1)}{\partial x_3}-\frac{\partial(h_3A_3)}{\partial x_1}\right)\right)\end{array}\right]\right\}\vec{e}_3$
Biharmonic operator, ∇^4	$\nabla^4 = \frac{1}{\mathcal{J}}\frac{\partial}{\partial x^k}\left\{\mathcal{J}g^{kl}\frac{\partial}{\partial x^l}\underbrace{\left[\frac{1}{\mathcal{J}}\frac{\partial}{\partial x^i}\left(\mathcal{J}g^{ij}\frac{\partial}{\partial x^j}\right)\right]}_{\nabla^2}\right\} = \frac{1}{h_1h_2h_3}\left\{\begin{array}{l}\frac{\partial}{\partial x_1}\left[\frac{h_2h_3}{h_1}\frac{\partial}{\partial x_1}\left[\frac{1}{h_1h_2h_3}\frac{\partial}{\partial x_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial x_1}\right)\right]\right]\\+\frac{\partial}{\partial x_2}\left[\frac{h_1h_3}{h_2}\frac{\partial}{\partial x_2}\left[\frac{1}{h_1h_2h_3}\frac{\partial}{\partial x_2}\left(\frac{h_1h_3}{h_2}\frac{\partial}{\partial x_2}\right)\right]\right]\\+\frac{\partial}{\partial x_3}\left[\frac{h_1h_2}{h_3}\frac{\partial}{\partial x_3}\left[\frac{1}{h_1h_2h_3}\frac{\partial}{\partial x_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial x_3}\right)\right]\right]\end{array}\right\}$

Modern engineering software tools provide facilities to include curvilinear coordinate systems for computer modeling. For example, COMSOL Multiphysics® has a *curvilinear coordinate interface* for defining object orientation in an arbitrary system [12].

CHAPTER 15

RIGID BODY ROTATION: EULER ANGLES, QUATERNIONS, AND ROTATION MATRIX

In Chapter 12, we discussed Cartesian tensors and rotation of coordinate system using Rodrigues' formula (see Equation 12.6). For practical applications, mainly rigid body rotation, in this chapter we would like to expand on this topic in relation to *Euler angles* and *quaternions* methods.

A rigid body transformation can be composed of two parts: a) translation and b) rotation. The translation is usually referred to a vector connecting the origin of the coordinate system to the centre of mass. Therefore, we can always refer the rotation to the instantaneous center of mass and focus on the rotation for calculation and analysis of rigid body motion.

In general, there are at least eight methods to represent rotation [13]. Among these three inter-related methods commonly used for calculating rigid body orientation after it goes through possible rotations in a 3D space. These are: 1) rotation matrix, 2) single axis-angle, and 3) quaternions. The first two methods are discussed in Chapter 12 and expanded in this chapter with the inclusion of *Euler angles* method. The third method is more robust and with less limitations compared to the other methods in terms of practical applications and is new to this second edition.

When dealing with rotation of a rigid body, several confusions may arise in terms of definition of rotation matrix and rotation variables. To avoid these ambiguities, identifying two definitions is critical. First, the assumed transformation should be clear by the way the rotation matrix (see Equation 12.7) is meant to be, i.e., it is meant to relate the coordinates obtained after rotation to the original ones or vice versa. Second, how the rotation itself is performed, i.e., is it that the coordinates are rotated and quantities, like a vector, is kept fixed and we seek to find its new coordinates with reference to the rotated ones or is it that the quantities, like a vector, are rotated and the coordinates are kept fixed and we would like to calculate the coordinates of the rotated vector with reference to the fix-kept coordinates.

In this book, we resolve these confusing issues by following the convention described in Chapter 12, i.e., rotation matrix relates the known/given coordinates prior to the rotation to those resulted from the rotation through the rotation matrix ($y'_i = R_{ij} y_j$). Therefore, we assign a passive convention, see next section. Assuming this definition then the second definition- so-called active and passive rotations [14]- is discussed in next section. Readers should note that some authors/references may have different definition assumed (sometimes implicitly) for the rotation matrix (e.g., $y_j = R_{ji} y'_i$, or using the transpose of the rotation matrix considered in this book).

Also, readers should note that an important fundamental property of all methods mentioned above is that the rotation procedure reduces to *the rotation of one vector about another vector*. This can be shown by finding the equivalent single axis and angle for any combinations of rotations [14], see Chapter 12.

Examples of practical applications of the methods mentioned above are: rigid body rotation and orientation, computer graphics and animation, aircraft maneuver control, robot arm control, signal processing, quantum mechanics, special and general relativity-among others, [15], [16].

15.1 ACTIVE AND PASSIVE ROTATIONS

Following Section 12.1, let's assume, without losing generality, a 2D coordinate system $(y_1, y_2) \equiv (x, y)$ being transformed to $(y'_1, y'_2) \equiv (x', y')$ through rotation about the axis $y_3 \equiv z$ with unit vector $\vec{n} \equiv n_i = (0,0,1)$ by an angle θ, according to R.H.R. Therefore, using the rotation matrix R_{ij} (see Equation 12.7) we have $\begin{Bmatrix} x' \\ y' \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$ and $z' = z$. Readers should note that the rotation matrix given here relates the rotated coordinates to the existing/current ones. Now if we have a vector $\vec{V}=\left(V_x,V_y\right)$ given in $x-y$ system and making angle φ with positive x axis then its components in $x'-y'$ system, in terms of V_x and V_y, are given as $\begin{Bmatrix} V_{x'} \\ V_{y'} \end{Bmatrix}=\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{Bmatrix} V_x \\ V_y \end{Bmatrix}$. This type of transformation for which the vector is kept fixed and the coordinate system rotates is called *passive rotation*. We can also perform the opposite operation by rotating the vector and keep the original coordinate system fixed. Therefore, we keep the system $x-y$ fixed and rotate the vector $\vec{V}$ by an angle equal to $-\theta$ about the same rotation axis (i.e., $n_i=(0,0,1)$) to get the vector $\vec{V}'$. This type of transformation for which the vector is rotated, and the coordinate system is kept fixed is called *active rotation*. See Figure 15.1. The components of the rotated vector are $\vec{V}'=\begin{Bmatrix} V_x' \\ V_y' \end{Bmatrix}$ and are equivalent to $\begin{Bmatrix} V_{x'} \\ V_{y'} \end{Bmatrix}$ in their magnitudes. In other words, passive rotation through an angle θ is equivalent to corresponding active rotation through an angle equal to $-\theta$. For 3D rotations, we can expand the relationship between passive and active rotations by using the properties of rotation matrix, see Section 12.1, or

$$R_{ij}\left(\vec{n},\theta\right)=R_{ji}\left(\vec{n},-\theta\right) \qquad 15.1$$

Equation 15.1 states that the rotation matrix for a given angle and axis of rotation is equal to its transpose when the rotation angle is in opposite direction but about the same rotation axis. As mentioned in this book, we consider R_{ij}, as given by Equation 12.7, for passive rotations and use its transpose (or inverse) for active rotations.

15.1.1 Example: Active and Passive Rotations-Numerical Example

Vector $\vec{V}=\left(V_x,V_y\right)=(3,4)$ is given in Cartesian coordinate system (x,y), as shown in Figure 15.1. Show that passive and active rotations provide equivalent answers. The absolute value of angle of rotation is $\theta=30^{\circ}$ and rotation is about positive z-axis.

Solution:

The magnitude of the vector $\vec{V}$ is $\left|\vec{V}\right| = \sqrt{3^2 + 4^2} = 5$ and it makes an angle $\varphi = \tan^{-1}\left(\dfrac{4}{3}\right) = 53.13°$ with the x-axis. For passive rotation, we keep the vector $\vec{V}$ fixed and rotate the coordinate system (x,y) by 30° in the counter clockwise direction to get the resulted coordinate system (x',y'). Therefore, the rotation matrix is $\begin{bmatrix} \cos 30 & \sin 30 \\ -\sin 30 & \cos 30 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$. The components of the vector $\vec{V}$ in the (x',y') system are $\begin{Bmatrix} V_{x'} \\ V_{y'} \end{Bmatrix}_{passive} = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} \begin{Bmatrix} 3 \\ 4 \end{Bmatrix} = \begin{Bmatrix} 4.598 \\ 1.964 \end{Bmatrix}$.

For active rotation, we keep the coordinate system (x,y) fixed and rotate the vector by 30° in the clockwise direction (or -30°) to receive the vector $\overrightarrow{V'} = \left(V'_x, V'_y\right)$ that makes an angle $\varphi' = 53.13^\circ - 30^\circ = 23.13^\circ$ with the x-axis. Therefore, the components are $V'_x = 5\cos(23.13) = 4.598$ and $V'_y = 5\sin(23.13) = 1.964$, or $\begin{Bmatrix} V'_x \\ V'_y \end{Bmatrix}_{active} = \begin{Bmatrix} 4.598 \\ 1.964 \end{Bmatrix}$. Note that the rotation matrix follows the relation given in Equation 15.1, or $R_{ij}\left(30^\circ\right) = \begin{bmatrix} \cos 30 & \sin 30 \\ -\sin 30 & \cos 30 \end{bmatrix} = R_{ji}\left(-30^\circ\right) = \begin{bmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{bmatrix}^T$.

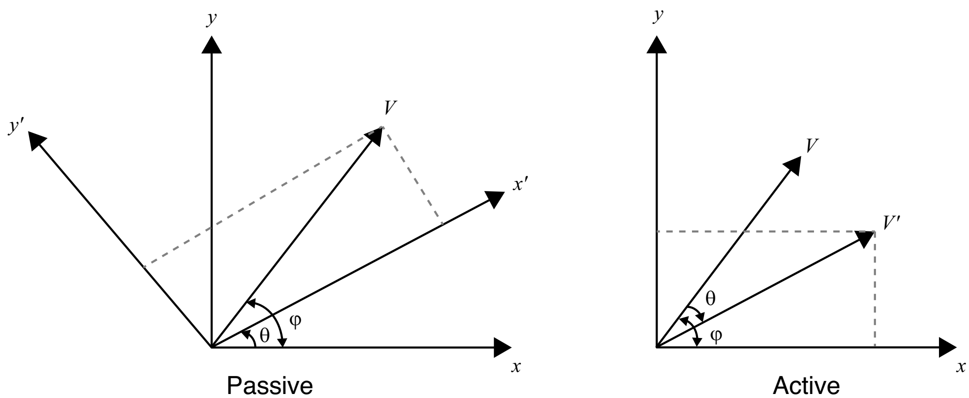


FIGURE 15.1 Passive (left) and active (right) transformations for a 2D coordinate system.

15.2 EULER ANGLES

We expand the discussion presented in the previous section by considering 3D rotation of a rigid body, for example an airplane. With reference to Chapter 12, we consider the following. Let's assume the airplane maneuvers in the sky, we can then relate its orientation, focusing on its rotation only, at any given time to its original orientation in a step-wise manner in two ways; 1) by relating its successive rotations to a fixed global Cartesian coordinate system or 2) by relating its orientations to its instantaneous body-attached coordinate system resulted from successive rotations. The latter is so-called *Intrinsic,* and the former is *Extrinsic* method. In this book, we use the intrinsic approach, [17], [18]. The extrinsic approach is more common for computer graphics applications.

The concept of describing orientation of a rigid body by a limited number of successive rotations goes back to Euler. His theorem, the Euler's rotation theorem [18], [19], [14] from a practical application point of view, states that any orientation of a rigid body can be calculated through a single rotation about a specified axis, with reference to its original orientation. This relates to what we showed and used, single axis-angle method, in Chapter 12, see Example 12.2.1. Further, Euler showed that the final orientation of a rigid body can be obtained through at most three successive rotations, so called *Euler angles*, [18], [20], [21]. The intrinsic method, as mentioned above, is mostly used for successive rotations.

There exist several definitions for the body-attached coordinate system in terms of axis orientations. Here we use the convention mostly used in engineering by defining positive x-axis (*roll*) direction towards the front/cockpit, y-axis (*pitch*) towards the right wing, and z-axis (*yaw*) downward towards the ground along with positive rotations about each axis. See Figure 15.2, for which R.H.R. applies.

Two immediate successive rotations around the same axis are not counted as two separate/independent rotations since we can equally have a single rotation equal to the algebraic sum of the two and hence the three-axis rotation reduces to two-axis rotation. Therefore, in 3D space, having (x,y,z) coordinates, we can have 12 $(= 3 \times 2 \times 2)$ combinations of Euler angles as shown in Table 15.1.

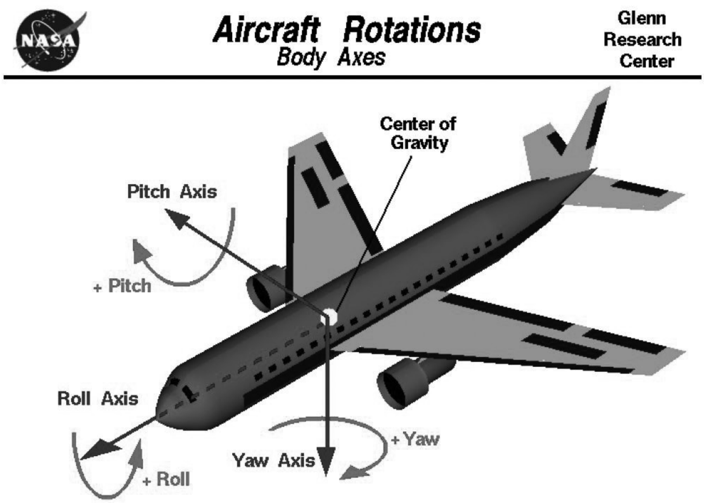


FIGURE 15.2 Body attached coordinates and corresponding Euler angles.

TABLE 15.1 Possible combination of Euler angles for rigid body rotations.

Roll group		Pitch group		Yaw group	
xyz (roll-pitch-yaw)	*xzy* (roll-yaw-pitch)	*yxz* (pitch-roll-yaw)	*yzx* (pitch-yaw-roll)	*zxy* (yaw-roll-pitch)	*zyx* (yaw-pitch-roll)
xyx (roll-pitch-roll)	*xzx* (roll-yaw-roll)	*yxy* (pitch-roll-pitch)	*yzy* (pitch-yaw-pitch)	*zxz* (yaw-roll-yaw)	*zyz* (yaw-pitch-yaw)

Few points should be noted about the order of executing Euler angles. As previously explained the rotation matrix related to each successive rotation should be pre-multiplied by the previous ones. Considering the order of rotations as the reference. This is a direct consequence of our definition for rotation matrix that transforms previous axes to the current rotated ones. Conversely, if one defines the transpose of the rotation matrix given by Equation 12.7 (i.e., transforming current/rotated coordinates to the original ones) then the sequence of rotations through Euler angles and their multiplications of their corresponding rotation matrices to generate the final rotation matrix should be performed in the same order. For example, according to our definition, see Equation 12.7, considering the combination *yxz* the order of rotation is *pitch* first, followed by *roll* and having *yaw* as the last rotation, or (pitch-roll-yaw). With this definition, *yxz* should be read in backward order, $\overleftarrow{yxz}$ to get the correct final rotation matrix resulted from multiplication of rotation matrices combination, see Section 12.2. Therefore, the related rotation matrices multiplication order is $\left[R_{\theta_3}\right]\left[R_{\theta_2}\right]\left[R_{\theta_1}\right]=\left[R_{\theta_1}\right]^T\left[R_{\theta_2}\right]^T\left[R_{\theta_3}\right]^T$, where θ_1 is the first angle of rotation (for example pitch), θ_2 second (for example roll), and θ_3 third (for example yaw). The resulted rotation matrix reads

$$[R]=\left[R_{\theta_3}\right]\left[R_{\theta_2}\right]\left[R_{\theta_1}\right] \tag{15.2}$$

In this book, we use the convention of index-order from 1 to 3 for successive rotations, i.e., first rotation expressed as θ_1 and so on. Once the rotation matrix $[R]$ is given, it is possible to calculate the corresponding Euler angles which are 12 sets, or mutations of 3 Euler angles corresponding to the rotation matrix, see Table 15.1.

As discussed in Chapter 12, we can use rotation matrix $[R]$ to calculate the single axis-angle rotation equivalent to three successive rotations by Euler angles. The single angle is given by Equation 12.8, repeated here for convenience, or $\cos\theta=\frac{R_{ii}-1}{2}=\frac{R_{11}+R_{22}+R_{33}}{2}-\frac{1}{2}$. Recall that R_{ii} is the trace of matrix $[R]$. The single axis of rotation is the eigenvalue of $[R]$, refer to Example 12.2.1. Another method, and may be a quicker one, for calculating the components of a unit vector along the rotation axis, represented by the unit vector $\vec{n}\equiv n_k$, can be obtained by manipulating the diagonal elements of $[R]$. For example, considering element $R(1,1)=\cos\theta+(n_1)^2(1-\cos\theta)$, see Equation 12.7.

After substituting for $\cos\theta$ we receive $R(1,1)=\frac{R_{ii}-1}{2}+(n_1)^2\left(1-\frac{R_{ii}-1}{2}\right)$.

Factoring out n_1 gives, after some manipulation, $|n_1|=\sqrt{\frac{2R(1,1)-R_{ii}+1}{3-R_{ii}}}$.

Therefore, for component k of the vector $\vec{n}=(n_1,n_2,n_3)$, parallel to the rotation axis, we receive

$$|n_k| = \sqrt{\frac{2R(k,k) - R_{ii} + 1}{3 - R_{ii}}},\ k = 1,2,3 \qquad 15.3$$

As an example, using data given in Example 12.2.1, we receive the trace of the rotation matrix, $R_{ii} = 0$ and $R(k,k) = 0$. Using Equation 15.3, we receive $|n_k| = 1/\sqrt{3}$, or $\vec{n} = \pm\frac{1}{\sqrt{3}}(1,1,1)$. This is identical to the results obtained in Example 12.2.1.

15.2.1 Example: Euler Angles-Numerical Example

An airplane undergoes a *zyz* (yaw-pitch-yaw) maneuver. The first yaw angle is $\theta_1 = 30^\circ$, pitch angle $\theta_2 = 45^\circ$, and the final yaw angle is $\theta_3 = 15^\circ$. Calculate its final orientation in the sky with reference to the earth fixed coordinates and the equivalent single axis-angle rotation.

Solution:

We calculate the rotation matrices, using Equation 12.6 and multiply them using Equation 15.2.

First yaw rotation: the rotation is about z-axis with $n_i = (0,0,1)$ and $\theta_1 = 30^\circ$ which gives $[R_{\theta_1}] = \begin{bmatrix} \cos 30 & \sin 30 & 0 \\ -\sin 30 & \cos 30 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Pitch rotation: this rotation is about current y-axis (resulted from the previous rotation) with $n_i = (0,1,0)$ and $\theta_2 = 45^\circ$ which gives

$$[R_{\theta_2}] = \begin{bmatrix} \cos 45 & 0 & -\sin 45 \\ 0 & 1 & 0 \\ \sin 45 & 0 & \cos 45 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}.$$

Last yaw rotation: this rotation is about current z-axis (resulted from the immediate previous rotation) with $n_i = (0,0,1)$ and $\theta_3 = 15^\circ$ which gives

$$\left[R_{\theta_3}\right] = \begin{bmatrix} \cos 15 & \sin 15 & 0 \\ -\sin 15 & \cos 15 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}+1}{2\sqrt{2}} & \frac{\sqrt{3}-1}{2\sqrt{2}} & 0 \\ -\frac{\sqrt{3}-1}{2\sqrt{2}} & \frac{\sqrt{3}+1}{2\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For final rotation matrix, we first calculate $\left[R_{\theta_2}\right]\left[R_{\theta_1}\right] =$

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

. The final operation is to pre-multiply $\left[R_{\theta_3}\right]$ to this result, or

$$[R] = \begin{bmatrix} \frac{\sqrt{3}+1}{2\sqrt{2}} & \frac{\sqrt{3}-1}{2\sqrt{2}} & 0 \\ -\frac{\sqrt{3}-1}{2\sqrt{2}} & \frac{\sqrt{3}+1}{2\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3+\sqrt{2}+\sqrt{3}-\sqrt{6}}{8} & \frac{1+3\sqrt{2}+\sqrt{3}-\sqrt{6}}{8} & -\frac{1+\sqrt{3}}{4} \\ \frac{-3-\sqrt{2}+\sqrt{3}-\sqrt{6}}{8} & \frac{1+3\sqrt{2}-\sqrt{3}+\sqrt{6}}{8} & \frac{-1+\sqrt{3}}{4} \\ \frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\cong \begin{bmatrix} 0.4621 & 0.5657 & -0.6830 \\ -0.6415 & 0.7450 & 0.1830 \\ 0.6124 & 0.3536 & 0.7071 \end{bmatrix}.$$

The corresponding single rotation angle is, using Equation 12.8, $\theta = \cos^{-1}\left(\frac{\sqrt{2}}{2} - \frac{1}{4}\right) = 62.8^{\circ}$. The elements of the unit vector along the single rotation axis are obtained using 15.3, or $|n_1| = \sqrt{\frac{2 \times 0.4621 - 1.9142 + 1}{3 - 1.9142}} = 9.6 \times 10^{-2}$, $|n_2| = \sqrt{\frac{2 \times 0.745 - 1.9142 + 1}{3 - 1.9142}} = 0.7282$, and $|n_3| = \sqrt{\frac{2 \times 0.7071 - 1.9142 + 1}{3 - 1.9142}} =$ 0.6786. Answers are rounded. In other words, we can obtain the final orientation of the airplane by rotating it through an angle of 62.8° about an axis in the direction of $n_k = (0.096, 0.7282, 0.6786)$ with reference to the original coordinate system (x, y, z). The successive rotations are shown in Figure 15.3.

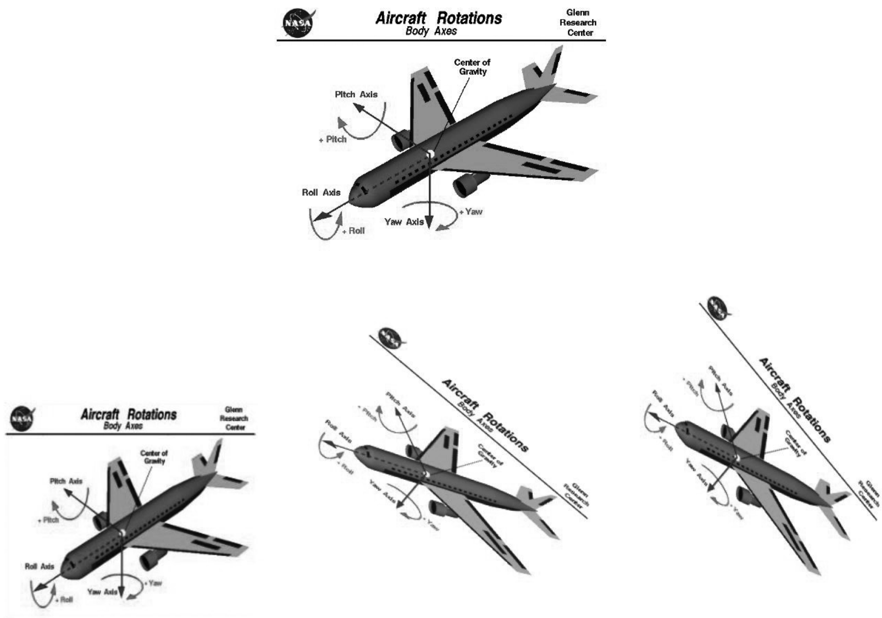


FIGURE 15.3 Successive rotation of an airplane, Euler angels left-to-right, yaw=30, pitch=45, yaw=15 degrees.

It seems useful to refer the readers to some animation tools for Euler angles. The following Apps, available through the following links, could be used for demonstrating the Euler angles.

"Controlling Airplane Flight" [22], "Euler Angles for Space Shuttle" [23], and "Euler Angles" [24].

15.3 CATEGORIZING EULER ANGLES

There are several ways possible to categorize Euler angles. With reference to Table 15.1, we observe that there are two sets of Euler angles combination. One set is those six combinations that contain repeated rotations (1st and 3rd) about similar (but not the same) axes (*xyx*, *xzx*, *yxy*, *yzy*, *zxz*, *zyz*), the so-called *proper Euler angles* and another set that has different axis of rotations for all three rotations (*xyz*, *xzy*, *yxz*, *yzx*, *zxy*, *zyx*), the so-called *Tait-Bryan angles*. The latter is commonly used in engineering, specifically in aerospace discipline [18], [19]. It is useful to collect all Euler angles types and their corresponding rotation matrices, as shown in Table 15.2 and Table 15.3. The corresponding single-axis rotation matrix of each case can be calculated as well, like those in Section 15.2.1.

TABLE 15.2 Proper Euler angles and rotation matrices.

unit vector-rotation axis		Roll: $n_i = (1,0,0)$			Pitch: $n_i = (0,1,0)$	Yaw: $n_i = (0,0,1)$
Rotation matrix-step		$\begin{bmatrix}1 & 0 & 0\\ 0 & \cos() & \sin()\\ 0 & -\sin() & \cos()\end{bmatrix}$			$\begin{bmatrix}\cos() & 0 & -\sin()\\ 0 & 1 & 0\\ \sin() & 0 & \cos()\end{bmatrix}$	$\begin{bmatrix}\cos() & \sin() & 0\\ -\sin() & \cos() & 0\\ 0 & 0 & 1\end{bmatrix}$
Successive rotation-angles		θ_1	θ_2	θ_3	Matrices multiplication for successive rotations	Rotation matrix
Proper Euler Angles	*xyx*	*roll*	*pitch*	*roll*	$\begin{bmatrix}1 & 0 & 0\\ 0 & \cos(\theta_3) & \sin(\theta_3)\\ 0 & -\sin(\theta_3) & \cos(\theta_3)\end{bmatrix}\begin{bmatrix}\cos(\theta_2) & 0 & -\sin(\theta_2)\\ 0 & 1 & 0\\ \sin(\theta_2) & 0 & \cos(\theta_2)\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos(\theta_1) & \sin(\theta_1)\\ 0 & -\sin(\theta_1) & \cos(\theta_1)\end{bmatrix}$	$= \mathbb{R}_{xyx}$
	xzx	*roll*	*yaw*	*roll*	$\begin{bmatrix}1 & 0 & 0\\ 0 & \cos(\theta_3) & \sin(\theta_3)\\ 0 & -\sin(\theta_3) & \cos(\theta_3)\end{bmatrix}\begin{bmatrix}\cos(\theta_2) & \sin(\theta_2) & 0\\ -\sin(\theta_2) & \cos(\theta_2) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos(\theta_1) & \sin(\theta_1)\\ 0 & -\sin(\theta_1) & \cos(\theta_1)\end{bmatrix}$	$= \mathbb{R}_{xzx}$
	yxy	*pitch*	*roll*	*pitch*	$\begin{bmatrix}\cos(\theta_3) & 0 & -\sin(\theta_3)\\ 0 & 1 & 0\\ \sin(\theta_3) & 0 & \cos(\theta_3)\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos(\theta_2) & \sin(\theta_2)\\ 0 & -\sin(\theta_2) & \cos(\theta_2)\end{bmatrix}\begin{bmatrix}\cos(\theta_1) & 0 & -\sin(\theta_1)\\ 0 & 1 & 0\\ \sin(\theta_1) & 0 & \cos(\theta_1)\end{bmatrix}$	$= \mathbb{R}_{yxy}$
	yzy	*pitch*	*yaw*	*pitch*	$\begin{bmatrix}\cos(\theta_3) & 0 & -\sin(\theta_3)\\ 0 & 1 & 0\\ \sin(\theta_3) & 0 & \cos(\theta_3)\end{bmatrix}\begin{bmatrix}\cos(\theta_2) & \sin(\theta_2) & 0\\ -\sin(\theta_2) & \cos(\theta_2) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos(\theta_1) & 0 & -\sin(\theta_1)\\ 0 & 1 & 0\\ \sin(\theta_1) & 0 & \cos(\theta_1)\end{bmatrix}$	$= \mathbb{R}_{yzy}$
	zxz	*yaw*	*roll*	*yaw*	$\begin{bmatrix}\cos(\theta_3) & \sin(\theta_3) & 0\\ -\sin(\theta_3) & \cos(\theta_3) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos(\theta_2) & \sin(\theta_2)\\ 0 & -\sin(\theta_2) & \cos(\theta_2)\end{bmatrix}\begin{bmatrix}\cos(\theta_1) & \sin(\theta_1) & 0\\ -\sin(\theta_1) & \cos(\theta_1) & 0\\ 0 & 0 & 1\end{bmatrix}$	$= \mathbb{R}_{zxz}$
	zyz	*yaw*	*pitch*	*yaw*	$\begin{bmatrix}\cos(\theta_3) & \sin(\theta_3) & 0\\ -\sin(\theta_3) & \cos(\theta_3) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos(\theta_2) & 0 & -\sin(\theta_2)\\ 0 & 1 & 0\\ \sin(\theta_2) & 0 & \cos(\theta_2)\end{bmatrix}\begin{bmatrix}\cos(\theta_1) & \sin(\theta_1) & 0\\ -\sin(\theta_1) & \cos(\theta_1) & 0\\ 0 & 0 & 1\end{bmatrix}$	$= \mathbb{R}_{zyz}$

where,

$$\mathbb{R}_{xyx} = \begin{bmatrix} \cos(\theta_2) & \sin(\theta_1)\sin(\theta_2) & -\cos(\theta_1)\sin(\theta_2) \\ \sin(\theta_2)\sin(\theta_3) & \cos(\theta_3)\cos(\theta_1) & \cos(\theta_3)\sin(\theta_1) \\ & -\sin(\theta_3)\cos(\theta_2)\sin(\theta_1) & +\sin(\theta_3)\cos(\theta_2)\cos(\theta_1) \\ \cos(\theta_3)\sin(\theta_2) & -\sin(\theta_3)\cos(\theta_1) & \cos(\theta_3)\cos(\theta_2)\cos(\theta_1) \\ & -\cos(\theta_3)\cos(\theta_2)\sin(\theta_1) & -\sin(\theta_3)\sin(\theta_1) \end{bmatrix}$$

Conversely; for a given single-axis rotation matrix the corresponding Euler angles are: $\theta_1 = \tan^{-1}\left[\frac{R(1,2)}{-R(1,3)}\right]$, $\theta_2 = \tan^{-1}\left[\frac{\sqrt{1-R^2(1,1)}}{R(1,1)}\right]$, and $\theta_3 = \tan^{-1}\left[\frac{R(2,1)}{R(3,1)}\right]$

$$\mathbb{R}_{xzx} = \begin{bmatrix} \cos(\theta_2) & \cos(\theta_1)\sin(\theta_2) & \sin(\theta_1)\sin(\theta_2) \\ -\sin(\theta_2)\cos(\theta_3) & -\sin(\theta_1)\sin(\theta_3) & \cos(\theta_1)\sin(\theta_3) \\ & +\cos(\theta_3)\cos(\theta_2)\cos(\theta_1) & +\sin(\theta_1)\cos(\theta_2)\cos(\theta_3) \\ \sin(\theta_2)\sin(\theta_3) & -\sin(\theta_1)\cos(\theta_3) & \cos(\theta_1)\cos(\theta_3) \\ & -\cos(\theta_1)\cos(\theta_2)\sin(\theta_3) & -\sin(\theta_3)\cos(\theta_2)\sin(\theta_1) \end{bmatrix}$$

Conversely; for a given single-axis rotation matrix the corresponding Euler angles are: $\theta_1 = \tan^{-1}\left[\frac{R(1,3)}{R(1,2)}\right]$, $\theta_2 = \tan^{-1}\left[\frac{\sqrt{1-R^2(1,1)}}{R(1,1)}\right]$, and $\theta_3 = \tan^{-1}\left[\frac{R(3,1)}{-R(2,1)}\right]$

$$\mathbb{R}_{yxy} = \begin{bmatrix} \cos(\theta_3)\cos(\theta_1) & \sin(\theta_2)\sin(\theta_3) & -\cos(\theta_3)\sin(\theta_1) \\ -\sin(\theta_3)\cos(\theta_2)\sin(\theta_1) & & -\cos(\theta_1)\cos(\theta_2)\sin(\theta_3) \\ \sin(\theta_1)\sin(\theta_2) & \cos(\theta_2) & \cos(\theta_1)\sin(\theta_2) \\ \sin(\theta_3)\cos(\theta_1) & -\sin(\theta_2)\cos(\theta_3) & \cos(\theta_3)\cos(\theta_2)\cos(\theta_1) \\ +\cos(\theta_3)\cos(\theta_2)\sin(\theta_1) & & -\sin(\theta_3)\sin(\theta_1) \end{bmatrix}$$

Conversely; for a given single-axis rotation matrix the corresponding Euler angles are: $\theta_1 = \tan^{-1}\left[\frac{R(2,1)}{R(2,3)}\right]$, $\theta_2 = \tan^{-1}\left[\frac{\sqrt{1-R^2(2,2)}}{R(2,2)}\right]$, and $\theta_3 = \tan^{-1}\left[\frac{R(1,2)}{-R(3,2)}\right]$

$$\mathbb{R}_{yzy} = \begin{bmatrix} -\sin(\theta_3)\sin(\theta_1) & \sin(\theta_2)\cos(\theta_3) & -\sin(\theta_3)\cos(\theta_1) \\ +\cos(\theta_3)\cos(\theta_2)\cos(\theta_1) & & -\cos(\theta_3)\cos(\theta_2)\sin(\theta_1) \\ -\cos(\theta_1)\sin(\theta_2) & \cos(\theta_2) & \sin(\theta_1)\sin(\theta_2) \\ \sin(\theta_1)\cos(\theta_3) & \sin(\theta_2)\sin(\theta_3) & -\sin(\theta_3)\cos(\theta_2)\sin(\theta_1) \\ +\cos(\theta_1)\cos(\theta_2)\sin(\theta_3) & & +\cos(\theta_3)\cos(\theta_1) \end{bmatrix}$$

Conversely; for a given single-axis rotation matrix the corresponding Euler angles are: $\theta_1 = \tan^{-1}\left[\frac{R(2,3)}{-R(2,1)}\right]$, $\theta_2 = \tan^{-1}\left[\frac{\sqrt{1-R^2(2,2)}}{R(2,2)}\right]$, and $\theta_3 = \tan^{-1}\left[\frac{R(3,2)}{R(1,2)}\right]$

$$\mathbb{R}_{zxz} = \begin{bmatrix} \cos(\theta_3)\cos(\theta_1) & \cos(\theta_3)\sin(\theta_1) & \sin(\theta_3)\sin(\theta_2) \\ -\sin(\theta_3)\cos(\theta_2)\sin(\theta_1) & +\sin(\theta_3)\cos(\theta_2)\cos(\theta_1) & \\ -\sin(\theta_3)\cos(\theta_1) & \cos(\theta_3)\cos(\theta_2)\cos(\theta_1) & \cos(\theta_3)\sin(\theta_2) \\ -\sin(\theta_1)\cos(\theta_2)\cos(\theta_3) & -\sin(\theta_3)\sin(\theta_1) & \\ \sin(\theta_1)\sin(\theta_2) & -\cos(\theta_1)\sin(\theta_2) & \cos(\theta_2) \end{bmatrix}$$

Conversely; for a given single-axis rotation matrix the corresponding Euler angles are: $\theta_1 = \tan^{-1}\left[\frac{R(3,1)}{-R(3,2)}\right]$, $\theta_2 = \tan^{-1}\left[\frac{\sqrt{1-R^2(3,3)}}{R(3,3)}\right]$, and $\theta_3 = \tan^{-1}\left[\frac{R(1,3)}{R(2,3)}\right]$

$$\mathbb{R}_{zyz} = \begin{bmatrix} -\sin(\theta_3)\sin(\theta_1) & \sin(\theta_3)\cos(\theta_1) & -\cos(\theta_3)\sin(\theta_2) \\ +\cos(\theta_3)\cos(\theta_2)\cos(\theta_1) & +\sin(\theta_1)\cos(\theta_2)\cos(\theta_3) & \\ -\sin(\theta_1)\cos(\theta_3) & -\sin(\theta_3)\cos(\theta_2)\ \sin(\theta_1) & \sin(\theta_3)\sin(\theta_2) \\ -\sin(\theta_3)\cos(\theta_2)\cos(\theta_1) & +\cos(\theta_3)\cos(\theta_1) & \\ \cos(\theta_1)\sin(\theta_2) & \sin(\theta_1)\sin(\theta_2) & \cos(\theta_2) \end{bmatrix}$$

Conversely; for a given single-axis rotation matrix the corresponding Euler angles are: $\theta_1 = \tan^{-1}\left[\frac{R(3,2)}{R(3,1)}\right]$, $\theta_2 = \tan^{-1}\left[\frac{\sqrt{1-R^2(3,3)}}{R(3,3)}\right]$, and $\theta_3 = \tan^{-1}\left[\frac{R(2,3)}{-R(1,3)}\right]$

TABLE 15.3 Tait-Bryan angles and rotation matrices.

unit vector-rotation axis		Roll: $n_i=(1,0,0)$			Pitch: $n_i=(0,1,0)$	Yaw: $n_i=(0,0,1)$
Rotation matrix-step		$\begin{bmatrix}1 & 0 & 0\\ 0 & \cos() & \sin()\\ 0 & -\sin() & \cos()\end{bmatrix}$			$\begin{bmatrix}\cos() & 0 & -\sin()\\ 0 & 1 & 0\\ \sin() & 0 & \cos()\end{bmatrix}$	$\begin{bmatrix}\cos() & \sin() & 0\\ -\sin() & \cos() & 0\\ 0 & 0 & 1\end{bmatrix}$
Successive rotation-angles		θ_1	θ_2	θ_3	Matrices multiplication for successive rotations	Rotation matrix
Tait-Bryan Angles	*xyz*	*roll*	*pitch*	*yaw*	$\begin{bmatrix}\cos(\theta_3) & \sin(3) & 0\\ -\sin(\theta_3) & \cos(\theta_3) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos(\theta_2) & 0 & -\sin(\theta_2)\\ 0 & 1 & 0\\ \sin(\theta_2) & 0 & \cos(\theta_2)\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos(\theta_1) & \sin(\theta_1)\\ 0 & -\sin(\theta_1) & \cos(\theta_1)\end{bmatrix}$	$=\mathbb{R}_{xyz}$
	xzy	*roll*	*yaw*	*pitch*	$\begin{bmatrix}\cos(\theta_3) & 0 & -\sin(\theta_3)\\ 0 & 1 & 0\\ \sin(\theta_3) & 0 & \cos(\theta_3)\end{bmatrix}\begin{bmatrix}\cos(\theta_2) & \sin(\theta_2) & 0\\ -\sin(\theta_2) & \cos(\theta_2) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos(\theta_1) & \sin(\theta_1)\\ 0 & -\sin(\theta_1) & \cos(\theta_1)\end{bmatrix}$	$=\mathbb{R}_{xzy}$
	yxz	*pitch*	*roll*	*yaw*	$\begin{bmatrix}\cos(\theta_3) & \sin(\theta_3) & 0\\ -\sin(\theta_3) & \cos(\theta_3) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos(\theta_2) & \sin(\theta_2)\\ 0 & -\sin(\theta_2) & \cos(\theta_2)\end{bmatrix}\begin{bmatrix}\cos(\theta_1) & 0 & -\sin(\theta_1)\\ 0 & 1 & 0\\ \sin(\theta_1) & 0 & \cos(\theta_1)\end{bmatrix}$	$=\mathbb{R}_{yxz}$
	yzx	*pitch*	*yaw*	*roll*	$\begin{bmatrix}1 & 0 & 0\\ 0 & \cos(\theta_3) & \sin(\theta_3)\\ 0 & -\sin(\theta_3) & \cos(\theta_3)\end{bmatrix}\begin{bmatrix}\cos(\theta_2) & \sin(\theta_2) & 0\\ -\sin(\theta_2) & \cos(\theta_2) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos(\theta_1) & 0 & -\sin(\theta_1)\\ 0 & 1 & 0\\ \sin(\theta_1) & 0 & \cos(\theta_1)\end{bmatrix}$	$=\mathbb{R}_{yzx}$
	zxy	*yaw*	*roll*	*pitch*	$\begin{bmatrix}\cos(\theta_3) & 0 & -\sin(\theta_3)\\ 0 & 1 & 0\\ \sin(\theta_3) & 0 & \cos(\theta_3)\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos(\theta_2) & \sin(\theta_2)\\ 0 & -\sin(\theta_2) & \cos(\theta_2)\end{bmatrix}\begin{bmatrix}\cos(\theta_1) & \sin(\theta_1) & 0\\ -\sin(\theta_1) & \cos(\theta_1) & 0\\ 0 & 0 & 1\end{bmatrix}$	$=\mathbb{R}_{zxy}$
	zyx	*yaw*	*pitch*	*roll*	$\begin{bmatrix}1 & 0 & 0\\ 0 & \cos(\theta_3) & \sin(\theta_3)\\ 0 & -\sin(\theta_3) & \cos(\theta_3)\end{bmatrix}\begin{bmatrix}\cos(\theta_2) & 0 & -\sin(\theta_2)\\ 0 & 1 & 0\\ \sin(\theta_2) & 0 & \cos(\theta_2)\end{bmatrix}\begin{bmatrix}\cos(\theta_1) & \sin(\theta_1) & 0\\ -\sin(\theta_1) & \cos(\theta_1) & 0\\ 0 & 0 & 1\end{bmatrix}$	$=\mathbb{R}_{zyx}$

where,

$$\mathbb{R}_{xyz} = \begin{bmatrix} \cos(\theta_2)\cos(\theta_3) & \cos(\theta_1)\sin(\theta_3) & \sin(\theta_3)\sin(\theta_1) \\ & +\cos(\theta_3)\sin(\theta_2)\sin(\theta_1) & -\cos(\theta_3)\sin(\theta_2)\cos(\theta_1) \\ -\cos(\theta_2)\sin(\theta_3) & -\sin(\theta_3)\sin(\theta_2)\sin(\theta_1) & \cos(\theta_3)\sin(\theta_1) \\ & +\cos(\theta_3)\cos(\theta_1) & +\sin(\theta_3)\sin(\theta_2)\cos(\theta_1) \\ \sin(\theta_2) & -\cos(\theta_2)\sin(\theta_1) & \cos(\theta_2)\cos(\theta_1) \end{bmatrix}$$

Conversely; for a given single-axis rotation matrix the corresponding Euler angles are: $\theta_1 = \tan^{-1}\left[\frac{-R(3,2)}{R(3,3)}\right]$, $\theta_2 = \tan^{-1}\left[\frac{R(3,1)}{\sqrt{1-R^2(3,1)}}\right]$, $\theta_3 = \tan^{-1}\left[\frac{-R(2,1)}{R(1,1)}\right]$

$$\mathbb{R}_{xzy} = \begin{bmatrix} \cos(\theta_3)\cos(\theta_2) & \sin(\theta_3)\sin(\theta_1) & -\sin(\theta_3)\cos(\theta_1) \\ & +\cos(\theta_3)\sin(\theta_2)\cos(\theta_1) & +\cos(\theta_3)\sin(\theta_2)\sin(\theta_1) \\ -\sin(\theta_2) & \cos(\theta_1)\cos(\theta_2) & \sin(\theta_1)\cos(\theta_2) \\ \sin(\theta_3)\cos(\theta_2) & -\cos(\theta_3)\sin(\theta_1) & \sin(\theta_3)\sin(\theta_2)\sin(\theta_1) \\ & +\cos(\theta_1)\sin(\theta_2)\sin(\theta_3) & +\cos(\theta_3)\cos(\theta_1) \end{bmatrix}$$

Conversely; for a given single-axis rotation matrix the corresponding Euler angles are: $\theta_1 = \tan^{-1}\left[\frac{R(2,3)}{R(2,2)}\right]$, $\theta_2 = \tan^{-1}\left[\frac{-R(2,1)}{\sqrt{1-R^2(2,1)}}\right]$, $\theta_3 = \tan^{-1}\left[\frac{R(3,1)}{R(1,1)}\right]$

$$\mathbb{R}_{yxz} = \begin{bmatrix} \cos(\theta_3)\cos(\theta_1) & \cos(\theta_2)\sin(\theta_3) & -\cos(\theta_3)\sin(\theta_1) \\ +\sin(\theta_3)\sin(\theta_2)\sin(\theta_1) & & +\sin(\theta_3)\sin(\theta_2)\cos(\theta_1) \\ -\sin(\theta_3)\cos(\theta_1) & \cos(\theta_2)\cos(\theta_3) & \sin(\theta_3)\sin(\theta_1) \\ +\cos(\theta_3)\sin(\theta_2)\sin(\theta_1) & & +\cos(\theta_3)\sin(\theta_2)\cos(\theta_1) \\ \sin(\theta_1)\cos(\theta_2) & -\sin(\theta_2) & \cos(\theta_1)\cos(\theta_2) \end{bmatrix}$$

Conversely; for a given single-axis rotation matrix the corresponding Euler angles are: $\theta_1 = \tan^{-1}\left[\frac{R(3,1)}{R(3,3)}\right]$, $\theta_2 = \tan^{-1}\left[\frac{-R(3,2)}{\sqrt{1-R^2(3,2)}}\right]$, and $\theta_3 = \tan^{-1}\left[\frac{R(1,2)}{R(2,2)}\right]$

$$\mathbb{R}_{yzx} = \begin{bmatrix} \cos(\theta_1)\cos(\theta_2) & \sin(\theta_2) & -\sin(\theta_1)\cos(\theta_2) \\ \sin(\theta_3)\sin(\theta_1) - \cos(\theta_3)\cos(\theta_2)\cos(\theta_1) & \cos(\theta_3)\cos(\theta_2) & \sin(\theta_3)\cos(\theta_1) + \cos(\theta_3)\sin(\theta_2)\sin(\theta_1) \\ \cos(\theta_3)\sin(\theta_1) + \cos(\theta_1)\sin(\theta_2)\sin(\theta_3) & -\sin(\theta_3)\cos(\theta_2) & -\sin(\theta_3)\sin(\theta_2)\sin(\theta_1) + \cos(\theta_3)\cos(\theta_1) \end{bmatrix}$$

Conversely; for a given single-axis rotation matrix the corresponding Euler angles are: $\theta_1 = \tan^{-1}\left[\frac{-R(1,3)}{R(1,1)}\right]$, $\theta_2 = \tan^{-1}\left[\frac{R(1,2)}{\sqrt{1-R^2(1,2)}}\right]$, and $\theta_3 = \tan^{-1}\left[\frac{-R(3,2)}{R(2,2)}\right]$

$$\mathbb{R}_{zxy} = \begin{bmatrix} \cos(\theta_1)\cos(\theta_3) - \sin(\theta_1)\sin(\theta_2)\sin(\theta_3) & \cos(\theta_3)\sin(\theta_1) + \sin(\theta_3)\sin(\theta_2)\cos(\theta_1) & -\cos(\theta_2)\sin(\theta_3) \\ -\sin(\theta_1)\cos(\theta_2) & \cos(\theta_1)\cos(\theta_2) & \sin(\theta_2) \\ \sin(\theta_3)\cos(\theta_1) + \sin(\theta_1)\sin(\theta_2)\cos(\theta_3) & -\cos(\theta_3)\sin(\theta_2)\cos(\theta_1) + \sin(\theta_1)\sin(\theta_3) & \cos(\theta_2)\cos(\theta_3) \end{bmatrix}$$

Conversely; for a given single-axis rotation matrix the corresponding Euler angles are: $\theta_1 = \tan^{-1}\left[\frac{-R(2,1)}{R(2,2)}\right]$, $\theta_2 = \tan^{-1}\left[\frac{R(2,3)}{\sqrt{1-R^2(2,3)}}\right]$, and $\theta_3 = \tan^{-1}\left[\frac{-R(1,3)}{R(3,3)}\right]$

$$\mathbb{R}_{zyx} = \begin{bmatrix} \cos(\theta_1)\cos(\theta_2) & \sin(\theta_1)\cos(\theta_2) & -\sin(\theta_2) \\ -\cos(\theta_3)\sin(\theta_1) & \cos(\theta_3)\cos(\theta_1) & \cos(\theta_2)\sin(\theta_3) \\ +\cos(\theta_1)\sin(\theta_2)\sin(\theta_3) & +\sin(\theta_3)\sin(\theta_2)\sin(\theta_1) & \\ \sin(\theta_3)\sin(\theta_1) & -\sin(\theta_3)\cos(\theta_1) & \cos(\theta_2)\cos(\theta_3) \\ +\cos(\theta_3)\sin(\theta_2)\cos(\theta_1) & +\sin(\theta_1)\sin(\theta_2)\cos(\theta_3) & \end{bmatrix}$$

Conversely; for a given single-axis rotation matrix the corresponding Euler angles are: $\theta_1 = \tan^{-1}\left[\dfrac{-R(1,2)}{R(1,1)}\right]$, $\theta_2 = \tan^{-1}\left[\dfrac{-R(1,3)}{\sqrt{1-R^2(1,3)}}\right]$, and $\theta_3 = \tan^{-1}\left[\dfrac{R(2,3)}{R(3,3)}\right]$

15.3.1 Example: Euler Angles- Yaw-Pitch-Yaw, Numerical example

Repeat the example given in Section 15.2.1 using relation $\mathbb{R}_{zyz}$, given in Table 15.2.

Solution: using Table 15.2, we have $\theta_1 = 30^\circ$ for first yaw rotation, $\theta_2 = 45^\circ$ for pitch rotation, and $\theta_1 = 15^\circ$ for third yaw rotation. Substituting in relation for $\mathbb{R}_{zyz}$ gives

$$\mathbb{R}_{zyz} = \begin{bmatrix} -\sin(15)\sin(30) & \sin(15)\cos(30) & -\cos(15)\sin(45) \\ +\cos(15)\cos(45)\cos(30) & +\sin(30)\cos(45)\cos(15) & \\ -\sin(30)\cos(15) & -\sin(15)\cos(45)\sin(30) & \sin(15)\sin(45) \\ -\sin(15)\cos(30)\cos(45) & +\cos(15)\cos(30) & \\ \cos(30)\sin(45) & \sin(30)\sin(45) & \cos(45) \end{bmatrix}$$

$\simeq \begin{bmatrix} 0.4621 & 0.5657 & -0.6830 \\ -0.6415 & 0.7450 & 0.1830 \\ 0.6124 & 0.3536 & 0.7071 \end{bmatrix}$. These results are identical with those from the example given in Section 15.2.1 Therefore, $\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} 0.4621 & 0.5657 & -0.6830 \\ -0.6415 & 0.7450 & 0.1830 \\ 0.6124 & 0.3536 & 0.7071 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$, where (x',y',z') are coordinates of a fixed given point in space with initial coordinates (x,y,z). In other words, the initially body-attached axes (x,y,z) orient themselves to the (x',y',z') after the airplane goes through the given successive yaw-pitch-yaw rotations.

15.4 GIMBAL LOCK-EULER ANGLES LIMITATION

For a wide range of applications, the Euler angles method provide us a useful tool for calculating a rigid body orientation. However, for certain values of rotation angles the process breaks down and ran into a mathematical "singularity" situation. This is usually referred to *gimbal lock*. This happens when the rigid body loses one of its three rotational degrees of freedom and rotation of two axes coincide when the rotation about the third remaining axis is some integer multiple of $\frac{\pi}{2}$. To see the mathematical explanation and implication of gimbal lock, we use one of the cases from Table 15.3, for example roll-pitch-yaw with final rotation matrix $\mathbb{R}_{xyz}$, repeated here for convenience.

$$\mathbb{R}_{xyz}=\begin{bmatrix}\cos(\theta_2)\cos(\theta_3) & \begin{matrix}\cos(\theta_1)\sin(\theta_3)\\ +\cos(\theta_3)\sin(\theta_2)\sin(\theta_1)\end{matrix} & \begin{matrix}\sin(\theta_3)\sin(\theta_1)\\ -\cos(\theta_3)\sin(\theta_2)\cos(\theta_1)\end{matrix}\\ -\cos(\theta_2)\sin(\theta_3) & \begin{matrix}-\sin(\theta_3)\sin(\theta_2)\sin(\theta_1)\\ +\cos(\theta_3)\cos(\theta_1)\end{matrix} & \begin{matrix}\cos(\theta_3)\sin(\theta_1)\\ +\sin(\theta_3)\sin(\theta_2)\cos(\theta_1)\end{matrix}\\ \sin(\theta_2) & -\cos(\theta_2)\sin(\theta_1) & \cos(\theta_2)\cos(\theta_1)\end{bmatrix}$$

Let the pitch angle be $\theta_2=\frac{\pi}{2}$. Therefore, we receive, after some trigonometric simplifications,

$\mathbb{R}_{xyz}=\begin{bmatrix}0 & \sin(\theta_1+\theta_3) & -\cos(\theta_1+\theta_3)\\ 0 & \cos(\theta_1+\theta_3) & \sin(\theta_1+\theta_3)\\ 1 & 0 & 0\end{bmatrix}$. Writing the transformed coordinates, we receive $\begin{Bmatrix}x'\\ y'\\ z'\end{Bmatrix}=\begin{bmatrix}0 & \sin(\beta) & -\cos(\beta)\\ 0 & \cos(\beta) & \sin(\beta)\\ 1 & 0 & 0\end{bmatrix}\begin{Bmatrix}x\\ y\\ z\end{Bmatrix}$. Where $\beta=\theta_1+\theta_3$.

Writing the system of equations gives $x' = y\sin(\beta) - z\cos(\beta)$, $y' = y\cos(\beta) + z\sin(\beta)$, and $z' = x$. Therefore, the rotated z-axis and x-axis are coincided, or so-called locked. Obviously, the roll angle θ_1 and yaw angle θ_3 can take on several different values but their changes are indistinguishable since angle β changes value if either of θ_1 or θ_3 changes. When pitch angle is equal to $\frac{\pi}{2}$ (or in general, some odd integer multiple of $\frac{\pi}{2}$) then the rigid body degrees of freedom is reduced from three to two, since $z' = x$ and yaw and roll rotations provide the same result in terms of the rigid body orientation. Similarly, other rotation matrices as listed in Table 15.2 and Table 15.3 can be used to show their corresponding gimbal lock orientation, [25].

It seems useful to follow the successive rotations step-by-step as well, for helping with understanding the gimbal lock. The first rotation is roll by angle θ_1 which results in orientation $\begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_1) & \sin(\theta_1) \\ 0 & -\sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$. Obviously, these leaves x-axis intact, $x_1 = x$ and rotates $y - z$ plane about x-axis. The second rotation is pitch by angle θ_2 which results in orientation $\begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix} = \begin{bmatrix} \cos(\theta_2) & 0 & -\sin(\theta_2) \\ 0 & 1 & 0 \\ \sin(\theta_2) & 0 & \cos(\theta_2) \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix}$. Obviously, these leaves y_1-axis intact, $y_2 = y_1$ and rotates $x_1 - z_1$ plane about y_1-axis. Now for the value of $\theta_2 = \frac{\pi}{2}$ we receive $\begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix}$, or $x_2 = -z_1$, $y_2 = y_1$, and $z_2 = x_1 = x$. The latter relation shows that original x-axis is coinciding with z_2 axis, or so-called gimbal-locked. The last rotation is yaw by angle θ_3 which results in orientation $\begin{Bmatrix} x_3 \\ y_3 \\ z_3 \end{Bmatrix} = \begin{bmatrix} \cos(\theta_3) & \sin(\theta_3) & 0 \\ -\sin(\theta_3) & \cos(\theta_3) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix}$. Obviously, these leaves z_3-axis intact, $z_3 = z_2 = x_1 = x$ and rotates $x_2 - y_2$ plane about z_2-axis.

Performing the matrix multiplication using all three matrices, produces

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{Bmatrix} x_3 \\ y_3 \\ z_3 \end{Bmatrix} = \begin{bmatrix} \cos(\theta_3) & \sin(\theta_3) & 0 \\ -\sin(\theta_3) & \cos(\theta_3) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_1) & \sin(\theta_1) \\ 0 & -\sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix},$$

or $\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} 0 & \sin(\theta_1+\theta_3) & -\cos(\theta_1+\theta_3) \\ 0 & \cos(\theta_1+\theta_3) & \sin(\theta_1+\theta_3) \\ 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$, or again showing the loss of one degrees of freedom out of three as $z' = x$, or gimbal lock.

To recover the lost degree of freedom due to gimbal lock situation we can perform manual maneuver, if/when possible to rotate the rigid body as smoothly as possible out of the gimbal lock position. This approach is not practical, and the control system should be programmed such that close to values of rotation angle that creates gimbal lock a warning sent out and users become aware of it. As shown in the animation video, cited above/below, we can use other more robust methods for removing gimbal lock. This method makes use of quaternions. We discuss the mathematics of quaternions and their applications in rigid body rotation and matrix transformation, in the following sections. Readers interested in discussion about gimbal lock can consult with references cited [14], [26], [18].

Readers may also find it helpful to consult with the following lecture series, specifically those ones related to rigid body kinematics [27]. Also using Wolfram Alpha, readers can visualize a given quaternion, along with some related computations.

An example for animation of a gimbal lock position, which could be helpful to visualize this phenomenon, can be found at references [28] and [29].

15.5 QUATERNIONS-APPLICATIONS FOR RIGID BODY ROTATION

A quaternion is defined as a 4D complex number. Mathematically, it is represented with a combination of a scalar and a vector-like complex number. Its development started when William R. Hamilton was visioning and working on extending the complex numbers to more than 2D space, a real and a complex numbers dimension. He first struggled adding one dimension to the complex number (i.e., defining it in 3D space) and ran into difficulties for performing basic algebraic operations, like multiplications, etc. For example, the product of two imaginary numbers *ij* is not defined in 3D space when having one real and two imaginary dimensions. Then he realized that adding an additional dimension to the complex numbers and defining it in 3D space would solve the problem, hence *Quaternions* are defined as complex numbers in 4D space, [30]. This newly discovered mathematical quantity, the quaternions, found to be very useful in several fields of science/physics and engineering such as kinematics of rigid body orientation, robotics, aerospace, computer graphics/animation and gaming software, relativity (General and Special), quantum mechanics. In this section, we discuss their definition, algebraic operations, and their applications for rigid body rotation [18], [21].

15.5.1 Definition-Quaternion

A quaternion can be defined as the combination of a scalar and a vector, as $q=\left(q_0,\vec{q}\right)$. The vector-like part is in principle defined as a 3D complex number when using their multiplication relation defined as given by Equation 15.4. It is well-known that in 1843 Hamilton, excited with his spark discovery, engraved this expression as a message on Broome Bridge in Dublin, Ireland [31], [32].

$$i^2 = j^2 = k^2 = ijk = -1 \qquad 15.4$$

Therefore, we can write a quaternion q as

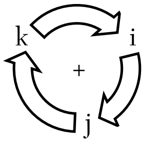
$$q = q_0 + iq_1 + jq_2 + kq_3 \equiv \left(q_0, q_1, q_2, q_3\right) \qquad 15.5$$

where, q_0 is a real number and remaining terms are complex numbers. For example, $2-i+3j+2k$. Using Equation 15.5, we can write complex numbers i, j, k as quaternions, or $i=\left(0,1,0,0\right)$, $j=\left(0,0,1,0\right)$, and $k=\left(0,0,0,1\right)$.

Note that the similarity between (i, j, k), here defined as complex numbers with those of unit vectors $\left(\vec{e}_1,\vec{e}_2,\vec{e}_3\right)\equiv\left(\vec{i},\vec{j},\vec{k}\right)$ that justifies the vector-like representation of a quaternion and hence to some extent simplifies the mathematical expressions resulted from algebraic manipulation of quaternions. The relations for the product of these complex quantities follows the cross-product of unit vectors, except for those of identical unit vectors (e.g., $i^2 = -1$, whereas $\vec{i} \times \vec{i} = 0$ and $\vec{i} \cdot \vec{i} = 1$, etc.). Please note $i^2 = ii$ is an expression representing arithmetic multiplication of a complex number i by itself, hence manipulation of quaternions is relatively easier than those of vectors.

All possible combinations of complex numbers products are listed in Table 15.4, with clockwise multiplication considered to be positive. From these relations we can conclude that these relations are not commutative, for example, $ij \neq ji = -ij$, etc.

TABLE 15.4 Hamilton relations, quaternions multiplication rule.

	$i = (0,1,0,0)$	$j = (0,0,1,0)$	$k = (0,0,0,1)$
i	$i^2 = -1$	$ij = k$	$ik = -j$
j	$ji = -k$	$j^2 = -1$	$jk = i$
k	$ki = j$	$kj = -i$	$k^2 = -1$

In terms of geometrical visualization, it is difficult to vision a 4D space since we are accustomed to visioning 3D space intuitively. It may help to imagine the surface of a hypersphere as representation of a surface defined in 4D space, like a 2D surface defined in a 3D space [21]. Readers may find it useful to watch the video clip cited in reference [33], demonstrating some animations of the geometry of a quaternion in terms of hyperspheres.

15.5.1.1 Example: Derivation of Hamilton Quaternion Relations

Using Equation 15.4, derive the identities given in Table 15.4. Recall that quaternions are not commutative.

Solution:

We start with $ijk = -1$ and multiply both sides by i to receive $\underbrace{i^2}_{-1} jk = -i$. Hence, $jk = i$. Similarly, $kj = -i$.

Now multiply both sides by j to receive $\underbrace{ji}_{-ij} jk = -j$. Hence, $-i \underbrace{j^2}_{-1} k = -j$ or $ik = -j$. Similarly, $ki = j$.

Now multiply both sides by k to receive $\underbrace{ki}_{-ik}jk = -k$. Hence, $i\underbrace{kj}_{-jk}k = k$ or $-ij\underbrace{k^2}_{-1} = k$. Hence, $ij = k$. Similarly, $ji = -k$.

15.5.2 Quaternions-Basic Algebraic Operations and Properties

We consider two quaternions, $q = q_0 + iq_1 + jq_2 + kq_3$ and $p = p_0 + ip_1 + jp_2 + kp_3$. The summation/subtraction operation on these quantities is straight forward and produces $q \pm p = (q_0 \pm p_0) + i(q_1 \pm p_1) + j(q_2 \pm p_2) + k(q_3 \pm p_3)$. Obviously, for more than two quaternions it can be extended to include as many as involved in the operation. However, the multiplication operation is not as straight forward. As a matter of fact, it was this operation that held Hamilton back for a while before he realized that a 4D space is required to properly define quaternions. The multiplication operation is as follow: $qp = (q_0 + iq_1 + jq_2 + kq_3)(p_0 + ip_1 + jp_2 + kp_3)$. Performing the term-by-term multiplications and using the relations given in Table 15.4, after some rearranging, we receive

$$\begin{aligned} qp = q_0p_0 - (q_1p_1 + q_2p_2 + q_3p_3) + q_0(ip_1 + jp_2 + kp_3) + p_0(iq_1 + jq_2 + kq_3) \\ +i(q_2p_3 - q_3p_2) + j(q_3p_1 - q_1p_3) + k(q_1p_2 - q_2p_1) \end{aligned} \tag{15.6}$$

This is a long expression and some simplifications is justified. We can make use of properties of the vector-like part of the quaternion, $\vec{q} = iq_1 + jq_2 + kq_3$ and $\vec{p} = ip_1 + jp_2 + kp_3$, and their dot and cross products when the complex numbers i, j, k are interpreted as unit vectors. Or $\vec{q}\cdot\vec{p} = q_1p_1 + q_2p_2 + q_3p_3$ and $\vec{q}\times\vec{p} = i(q_2p_3 - q_3p_2) + j(q_3p_1 - q_1p_3) + k(q_1p_2 - q_2p_1)$. Substituting for these expressions back into multiplication result (Equation 15.6), we receive

$$qp = \underbrace{(q_0p_0 - \vec{q}\cdot\vec{p})}_{scalar} + \underbrace{(q_0\vec{p} + p_0\vec{q} + \vec{q}\times\vec{p})}_{vector-complex\ number} \tag{15.7}$$

Equation 15.7 clearly demonstrates that $qp \neq pq$, since $\vec{q}\times\vec{p} \neq \vec{p}\times\vec{q}$.

The author's caution is to note the dual interpretations of complex numbers i, j, k as unit vectors when the complex part of a quaternion is treated as a vector (for example $i^2 = -1$ but we treat as $\vec{i}\cdot\vec{i} = 1$ or $\vec{i}\times\vec{i} = 0$ for vector operation). Readers should note that simply the results of the product operations are the same, not necessarily the term-by-term operation, hence Equation 15.7 provides us with the same answer as that of Equation 15.6.

However, Equation 15.7 is presented in more compact memorable form of the quaternions' product expression.

It is useful to define conjugate of the quaternion, $q^* = (q_0 - \vec{q}) = q_0 - (iq_1 + jq_2 + kq_3)$. We can use Equation 15.7, to show that $qq^* = q^*q = q_0^2 + |\vec{q}|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2$. This quantity is equal to the square of the norm or magnitude of the quaternion $|q|$, or

$$|q| = \sqrt{qq^*} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} = |q^*| \qquad 15.8$$

A unit quaternion, $Q = \frac{q}{|q|} = Q_0 + iQ_1 + jQ_2 + kQ_3$ is defined as a quaternion with magnitude/norm equal to unity, with having $QQ^* = 1$. Therefore, for a given unit quaternion its inverse and conjugate are equal, $Q^{-1} = Q^*$. Also, since $q^*q = |q|^2$, see Equation 15.8, we can define the inverse of the quaternion as $q^{-1} = \frac{q^*}{|q|^2}$. Therefore, the division operation of two quaternions can be performed. This, however, has two forms since the quaternions are non-commutative. For example, the ratio $\frac{q}{p}$ (assuming $p \neq 0$) can be written as qp^{-1} or $p^{-1}q$. Users should define which quotient is desired for their related calculations.

15.5.2.1 Quaternions Algebraic Operations-Numerical Example

Given $q = 3 + i - 2j + k$ and $p = 2 - i + 2j + 3k$, compute $q + p$, qp, their corresponding conjugate, norm, inverse, and unit quaternion.

Solution:

$q + p = (3 + i - 2j + k) + (2 - i + 2j + 3k) = 5 + 4k$. For the product we calculate the dot and cross products of the corresponding vectors, or $\vec{q} \cdot \vec{p} = (1,-2,1) \cdot (-1,2,3) = -1 - 4 + 3 = -2$ and $\vec{q} \times \vec{p} = e_{mnl} q_n p_l = -8i - 4j$. Therefore, using Equation 15.7, we have $qp = [6 + 2] + [-3i + 6j + 9k + 2i - 4j + 2k - 8i - 4j] = 8 - 9i - 2j + 11k$. The conjugates are $q^* = 3 - (i - 2j + k)$ and $p^* = 2 - (-i + 2j + 3k)$. The norm $|q| = \sqrt{9 + 1 + 4 + 1} = \sqrt{15}$ and $|p| = \sqrt{4 + 1 + 4 + 9} = 3\sqrt{2}$.

The inverse $q^{-1} = \frac{q^*}{|q|^2} = \frac{3-i+2j-k}{15} = \frac{1}{5} - \frac{1}{15}i + \frac{2}{15}j - \frac{1}{15}k$ and $p^{-1} = \frac{p^*}{|p|^2} = \frac{2+i-2j-3k}{18} = \frac{1}{9} + \frac{1}{18}i - \frac{1}{9}j - \frac{1}{6}k$. The unit quaternions are, $Q = \frac{3+i-2j+k}{\sqrt{15}} = \frac{\sqrt{15}}{5} + \frac{\sqrt{15}}{15}i - \frac{2\sqrt{15}}{15}j + \frac{\sqrt{15}}{15}k$ and $P = \frac{2-i+2j+3k}{3\sqrt{2}} = \frac{\sqrt{2}}{3} - \frac{\sqrt{2}}{6}i + \frac{\sqrt{2}}{3}j + \frac{\sqrt{2}}{2}k$.

Having the algebraic rules defined for quaternions, in the following sections we present their application for coordinates system and rigid body rotation. Since quaternions are basically an extension to the 2D complex numbers, first we discuss the relation between 2D complex numbers and rotation matrix followed by similar relations for 4D quaternions.

15.5.3 Complex Numbers and Rotation Matrix

Complex numbers can be used to represent rotation of a rigid body when Euler formula is used. Euler formula relates an exponential complex number/variable to periodic sinusoidal functions [25], or

$$e^{i\alpha} = \cos\alpha + i\sin\alpha \tag{15.9}$$

where $i = \sqrt{-1}$. For example, in a 2D space, we can define a complex variable $z = z(x,y) = x + iy$ representing a vector drawn from coordinates' origin to the point z on the $x-y$ plane and making an angle $= \tan^{-1}\left(\frac{y}{x}\right)$, *phase/argument*, with the positive x-axis and having the magnitude $|z| = \sqrt{zz^*} = \sqrt{(x+iy)(x-iy)}$. Writing down the components, we get $x = |z|\cos\varphi$ and $y = |z|\sin\varphi$. Or $z = |z|(\cos\varphi + i\sin\varphi) = |z|e^{i\varphi}$. Now, we operate the unit vector $e^{i\theta}$ on z, to get $z' = e^{i\theta}|z|e^{i\varphi} = |z|e^{i(\varphi+\theta)}$. Note that the magnitude of the original vector is preserved, but its phase angle changes to $(\varphi+\theta)$. Expanding this expression, using Equation 15.9, we receive the components of transformed vector $z' = (x+iy)e^{i\theta} = x\cos\theta - y\sin\theta + i(x\sin\theta + y\cos\theta)$, as (omitting i) $\begin{Bmatrix} x' \\ y' \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$. This is identical to active rotation operation discussed in Section 15.1. Similarly, operating $e^{-i\theta}$ on z gives the equivalent of passive rotation operation.

15.5.4 Quaternions and Rotation

To extend the application of complex numbers to 3D rotations, we make use of the process/method mentioned in the previous section and include quaternions in the discussion. First, we note that an arbitrary vector, $\vec{V} = iV_1 + jV_2 + kV_3 = (V_1, V_2, V_3)$ can be considered as a quaternion whose real part is zero, i.e., *pure quaternions*. Similarly, a *real quaternion* is a quaternion whose vector part is null with non-zero real part. Now, we consider a unit quaternion $Q = Q_0 + \vec{Q}$. Therefore, $|Q| = Q_0^2 + |\vec{Q}|^2 = Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2 = 1$ which implies that we can define an angle β such that

$$\begin{cases} Q_0 = \cos\beta \\ |\vec{Q}| = \sin\beta \end{cases} \tag{15.10}$$

We also define a unit vector as $\vec{n} = \frac{\vec{Q}}{|\vec{Q}|} = \frac{\vec{Q}}{\sqrt{Q_1^2 + Q_2^2 + Q_3^2}}$ using the vector part (i.e., vector $\vec{Q}$) of the unit quaternion Q. Therefore, we can write the unit quaternion as $Q = \cos\beta + \vec{n}\sin\beta$.

Now, we can write as an extension to Euler formula, see Equation 15.9, the so-called De Moivre's formula for quaternion Q, [26]

$$Q = e^{\vec{n}\beta} = \cos\beta + \vec{n}\sin\beta \tag{15.11}$$

Note that $\vec{n}$ is a 3D imaginary number treated as a vector-like quantity calculated from the imaginary part of the unit quaternion. Using Equation 15.10, we can write $\beta = \tan^{-1}\left(\frac{\sqrt{Q_1^2 + Q_2^2 + Q_3^2}}{Q_0}\right)$ and since $\vec{n} = (n_1, n_2, n_3)$ is defined as a unit vector we have $n_1^2 + n_2^2 + n_3^2 = 1$.

Recall that the rotation matrix/operation preserves the magnitude of the vector which goes under rotation. In other words, magnitude of a vector under rotation is invariant (see Sections 12.2 and 15.5.3). Therefore, for using quaternions in rotation operation we look for an operator which is a combination of relevant quaternion and maintains the magnitude of the vector on which it operates. The quaternion operation on a vector is done in a two-step process, using a unit quaternion and its inverse/conjugate to make sure that the result is a vector with preserved magnitude. For this purpose, we use the unit quaternion Q and form the combination $Q\vec{V}Q^* \left(= Q\vec{V}Q^{-1}\right)$ to transform vector $\vec{V}$ to $\vec{V}'$ by rotation, or

$$\vec{V}' = Q\vec{V}Q^* = e^{\vec{n}\beta}\vec{V}e^{-\vec{n}\beta} \tag{15.12}$$

As mentioned, the reason that we use unit quaternion and its inverse/conjugate is simply because the combination $Q\vec{V}$, as much as it may intuitively seem the correct combination, it doesn't result in a new vector and nor maintaining the magnitude of $\vec{V}$, under rotation operation. This can be seen using Equation 15.7 and writing $\vec{V}$ as a pure quaternion. Or $Q\vec{V} = (Q_0, Q_1, Q_2, Q_3)(0, V_1, V_2, V_3) = -\vec{Q}\cdot\vec{V} + Q_0\vec{V} + \vec{Q}\times\vec{V}$, which is clearly a quaternion quantity and not a vector due to having a scalar part (i.e. $\vec{Q}\cdot\vec{V}$).

Now, after expanding Equation 15.12, using Equation 15.11, we have $Q\vec{V}Q^* = (\cos\beta + \vec{n}\sin\beta)\vec{V}(\cos\beta - \vec{n}\sin\beta) = (\vec{V}\cos\beta + \vec{n}\vec{V}\sin\beta)(\cos\beta - \vec{n}\sin\beta) = \vec{V}\cos^2\beta - \vec{n}\vec{V}\vec{n}\sin^2\beta + (\vec{n}\vec{V} - \vec{V}\vec{n})\sin\beta\cos\beta$. But using vector identities, we can write $\vec{n}\vec{V}\vec{n} = \vec{V} - 2(\vec{n}\cdot\vec{V})\vec{n}$ and $\vec{n}\vec{V} - \vec{V}\vec{n} = 2\vec{n}\times\vec{V}$. After substituting we receive, in terms of double angle 2β,

$$Q\vec{V}Q^* = \vec{V}\cos 2\beta + (1-\cos 2\beta)(\vec{n}\cdot\vec{V})\vec{n} + (\vec{n}\times\vec{V})\sin 2\beta$$

Therefore, we can write Equation 15.12 in index notation as (note that in Equation 15.13 subscripts i, j, k are simply indices, ranging from 1 to 3 and should not be confused with the imaginary numbers)

$$V_i' = V_i\cos 2\beta + (1-\cos 2\beta)n_i n_j V_j + e_{ijk}n_j V_k \sin 2\beta \tag{15.13}$$

Writing Equation 15.13 in matrix form, let $\theta = 2\beta$, produces

$$\begin{Bmatrix} V_1' \\ V_2' \\ V_3' \end{Bmatrix} = \begin{bmatrix} \cos\theta + n_1 n_1(1-\cos\theta) & n_1 n_2(1-\cos\theta) - n_3\sin\theta & n_1 n_3(1-\cos\theta) + n_2\sin\theta \\ n_1 n_2(1-\cos\theta) + n_3\sin\theta & \cos\theta + n_2 n_2(1-\cos\theta) & n_2 n_3(1-\cos\theta) - n_1\sin\theta \\ n_1 n_3(1-\cos\theta) - n_2\sin\theta & n_2 n_3(1-\cos\theta) + n_1\sin\theta & \cos\theta + n_3 n_3(1-\cos\theta) \end{bmatrix} \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix} \tag{15.14}$$

This is identical to using Rodrigues equation when rotation angle is measured in the opposite direction, or the transpose of rotation matrix given in Equation 12.7. In other words, the operation performed according to Equations 15.12 or 15.14 gives active rotation of vector $\vec{V}$ through angle $\theta = 2\beta$ about unit vector $\vec{n}$. Therefore, the quaternions should be written as half-angle of the rotation angle (i.e., $\beta = \theta/2$), Or

$$\begin{cases} \theta_{rotation} = (2\beta)_{quaternion} \Big| = 2\tan^{-1}\left(\dfrac{|\vec{Q}|}{Q_0}\right) \\ \vec{n} = \dfrac{\vec{Q}}{\sqrt{Q_1^2 + Q_2^2 + Q_3^2}} \end{cases} \tag{15.15}$$

Readers may recall that for active rotation we keep the coordinate system fixed and rotate the vector, see Section 15.1. From the above-mentioned derivation, we can conclude the following theorem (note that $Q^{-1} = Q^*$):

For any given vector $\vec{V}$ and unit quaternion $Q = Q_0 + \vec{Q}$, vector $Q\vec{V}Q^{-1}$ is equivalent to the active rotation of the vector $\vec{V}$ through an angle $\theta = 2\tan^{-1}\left(\dfrac{|\vec{Q}|}{Q_0}\right)$ about an axis parallel to $\vec{Q}$ $\left(\text{or equivalently the unit vector } \vec{n} = \dfrac{\vec{Q}}{|\vec{Q}|}\right)$

Similarly, it can be shown that the combination

$$Q^*\vec{V}Q = e^{-\vec{n}\beta}\vec{V}e^{\vec{n}\beta} \tag{15.16}$$

is equivalent to a passive rotation of vector $\vec{V}$.

Two points worth mentioning, about quaternions and rotations. 1) It takes two quaternions, the unit quaternion itself and its inverse/conjugate, to perform a rotation for a rigid body. Each quaternion carries half of the total angle of rotation, i.e., $\beta = \dfrac{\theta}{2}$. 2) Further transformation follows the same rule. For example, the vector $Q^*\vec{V}Q$ can be transformed again using unit quaternion P representing a second rotation, or

$$P^*\underbrace{\left(Q^*\vec{V}Q\right)}_{1^{st}\ rotation}P = \underbrace{(QP)^*\,\vec{V}\,(QP)}_{2^{nd}\ rotation} = W^*\vec{V}W \tag{15.17}$$

As seen from Equation 15.17, we can combine the related two quaternions as their product $W = QP$ and perform the operation on the desired vector as a single rotation. The first rotation corresponds to Q and the second one to P. Similarly, for an active rotation we can write the following relation for two successive rotations,

$$P\underbrace{\left(Q\vec{V}Q^*\right)}_{1^{st}\ rotation}P^* = \underbrace{(PQ)\vec{V}(PQ)^*}_{2^{nd}\ rotation} = \mathbb{W}\vec{V}\mathbb{W}^* \tag{15.18}$$

Example: Show that following theorem is correct.

For any given vector $\vec{V}$ and unit quaternion $Q = Q_0 + \vec{Q}$, components of vector $Q^{-1}\vec{V}Q$ are equivalent to those of vector $\vec{V}$ in the rotated coordinates (i.e., passive rotation) by an angle $\theta = 2tan^{-1}\left(\frac{|\vec{Q}|}{Q_0}\right)$ about an axis parallel to the vector $\vec{Q}$ $\left(\text{or equivalently unit vector } \vec{n} = \frac{\vec{Q}}{|\vec{Q}|}\right)$

Solution:

Note that quaternions should be written as half of the total rotation angle, or $\beta = \frac{\theta}{2}$, $Q^{-1}\vec{V}Q = (\cos\beta - \vec{n}\sin\beta)\vec{V}(\cos\beta + \vec{n}\sin\beta) = \left(\vec{V}\cos\beta - \vec{n}\vec{V}\sin\beta\right)(\cos\beta + \vec{n}\sin\beta) = \vec{V}\cos^2\beta - \vec{n}\vec{V}\vec{n}\sin^2\beta - \left(\vec{n}\vec{V} - \vec{V}\vec{n}\right)\sin\beta\cos\beta$. However, using vector identities, we can write $\vec{n}\vec{V}\vec{n} = \vec{V} - 2\left(\vec{n}\cdot\vec{V}\right)\vec{n}$ and $\vec{n}\vec{V} - \vec{V}\vec{n} = 2\vec{n}\times\vec{V}$. Substituting, we obtain, in terms of double angle 2β, $Q^*\vec{V}Q = \vec{V}\cos 2\beta + (1-\cos 2\beta)\left(\vec{n}\cdot\vec{V}\right)\vec{n} - \left(\vec{n}\times\vec{V}\right)\sin 2\beta$. Or in index notation form we have (note that in Equations 15.13 and 15.19 subscripts i, j, k are simply indices range from 1 to 3 and should not be confused with the imaginary numbers)

$$V_{x_i'} = V_i\cos 2\beta + (1-\cos 2\beta)n_i n_j V_j - e_{ijk}n_j V_k \sin 2\beta \tag{15.19}$$

Writing Equation 15.19 in matrix form, let $\theta = 2\beta$, produces

$$\begin{Bmatrix} V_{x'} \\ V_{y'} \\ V_{z'} \end{Bmatrix} = \begin{bmatrix} \cos\theta + n_1 n_1 (1-\cos\theta) & n_1 n_2 (1-\cos\theta) + n_3 \sin\theta & n_1 n_3 (1-\cos\theta) - n_2 \sin\theta \\ n_1 n_2 (1-\cos\theta) - n_3 \sin\theta & \cos\theta + n_2 n_2 (1-\cos\theta) & n_2 n_3 (1-\cos\theta) + n_1 \sin\theta \\ n_1 n_3 (1-\cos\theta) + n_2 \sin\theta & n_2 n_3 (1-\cos\theta) - n_1 \sin\theta & \cos\theta + n_3 n_3 (1-\cos\theta) \end{bmatrix} \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix} \tag{15.20}$$

This is identical to the rotation matrix given in Equation 12.7. In other words, the operation given by $Q^{-1}\vec{V}Q$ gives the components of vector $\vec{V}$ when coordinates are rotated through angle $\theta = 2\beta$ about unit vector $\vec{n}$, see Equation 15.15. Readers may recall that for passive rotation we keep the vector fixed and rotate the coordinate system, see Section 15.1.

Example: Rotation using quaternions-numerical example

Consider the rotation of vector $\vec{V} = \vec{i} = (1,0,0)$ by an angle $\theta = \dfrac{2\pi}{3}$ about an axis in the direction $\vec{r} = (1,1,1)$. Find the transformed vector assuming active and passive rotations using following methods:

a. Rodrigues' formula

b. quaternions

Solution:

a. The unit vector related to the rotation axis is $\vec{n} = \dfrac{\vec{r}}{|\vec{r}|} = \left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$.

The transformation matrix is, using Equations 15.13 or 12.7, for active transformation (vector rotates and coordinates are fixed) gives (note that $\theta = -\dfrac{2\pi}{3}$ for active rotation, when plugged in Equation 12.7)

$$\begin{Bmatrix} V_1' \\ V_2' \\ V_3' \end{Bmatrix} =$$

$$\begin{bmatrix} \cos\frac{2\pi}{3} + \frac{1}{3}\left(1-\cos\frac{2\pi}{3}\right) & \frac{1}{3}\left(1-\cos\frac{2\pi}{3}\right) - \frac{1}{\sqrt{3}}\sin\frac{2\pi}{3} & \frac{1}{3}\left(1-\cos\frac{2\pi}{3}\right) + \frac{1}{\sqrt{3}}\sin\frac{2\pi}{3} \\ \frac{1}{3}\left(1-\cos\frac{2\pi}{3}\right) + \frac{1}{\sqrt{3}}\sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} + \frac{1}{3}\left(1-\cos\frac{2\pi}{3}\right) & \frac{1}{3}\left(1-\cos\frac{2\pi}{3}\right) - \frac{1}{\sqrt{3}}\sin\frac{2\pi}{3} \\ \frac{1}{3}\left(1-\cos\frac{2\pi}{3}\right) - \frac{1}{\sqrt{3}}\sin\frac{2\pi}{3} & \frac{1}{3}\left(1-\cos\frac{2\pi}{3}\right) + \frac{1}{\sqrt{3}}\sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} + \frac{1}{3}\left(1-\cos\frac{2\pi}{3}\right) \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

Or

$$\begin{Bmatrix} V_1' \\ V_2' \\ V_3' \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$. Hence active rotation of vector $\vec{i}$ gives vector $\vec{j}$. Similarly, the passive rotation (coordinates rotate and vector is fixed) gives (note that $\theta = \frac{2\pi}{3}$ for passive rotation, when plugged in Equation 12.7)

$$\begin{Bmatrix} V_{x'} \\ V_{y'} \\ V_{z'} \end{Bmatrix} =$$

$$\begin{bmatrix} \cos\frac{2\pi}{3} + \frac{1}{3}\left(1-\cos\frac{2\pi}{3}\right) & \frac{1}{3}\left(1-\cos\frac{2\pi}{3}\right) + \frac{1}{\sqrt{3}}\sin\frac{2\pi}{3} & \frac{1}{3}\left(1-\cos\frac{2\pi}{3}\right) - \frac{1}{\sqrt{3}}\sin\frac{2\pi}{3} \\ \frac{1}{3}\left(1-\cos\frac{2\pi}{3}\right) - \frac{1}{\sqrt{3}}\sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} + \frac{1}{3}\left(1-\cos\frac{2\pi}{3}\right) & \frac{1}{3}\left(1-\cos\frac{2\pi}{3}\right) + \frac{1}{\sqrt{3}}\sin\frac{2\pi}{3} \\ \frac{1}{3}\left(1-\cos\frac{2\pi}{3}\right) + \frac{1}{\sqrt{3}}\sin\frac{2\pi}{3} & \frac{1}{3}\left(1-\cos\frac{2\pi}{3}\right) - \frac{1}{\sqrt{3}}\sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} + \frac{1}{3}\left(1-\cos\frac{2\pi}{3}\right) \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$. Hence passive rotation of vector $\vec{i}$ gives vector $\vec{k}$.

b. When quaternions are used for rotation note that half-angle should be used $\left(\beta = \theta / 2 = \frac{\pi}{3}\right)$. The unit quaternion reads, using Equation 15.11, $Q = \cos\frac{\pi}{3} + \vec{n}\sin\frac{\pi}{3} = \frac{1}{2} + \left(\frac{i}{\sqrt{3}} + \frac{j}{\sqrt{3}} + \frac{k}{\sqrt{3}}\right)\frac{\sqrt{3}}{2} = \frac{1}{2} + \frac{i}{2} + \frac{j}{2} + \frac{k}{2}$. Hence, $Q_0 = \frac{1}{2}$ and $\vec{Q} = \frac{i}{2} + \frac{j}{2} + \frac{k}{2}$. For active rotation, we have $Q\vec{V}Q^* = \left(\frac{1}{2} + \frac{i}{2} + \frac{j}{2} + \frac{k}{2}\right)(i)\left(\frac{1}{2} - \frac{i}{2} - \frac{j}{2} - \frac{k}{2}\right) = \left(-\frac{1}{2} + \frac{i}{2} - \frac{k}{2} + \frac{j}{2}\right)\left(\frac{1}{2} - \frac{i}{2} - \frac{j}{2} - \frac{k}{2}\right) = j$. Similarly, for passive rotation we have $Q^*\vec{V}Q = \left(\frac{1}{2} - \frac{i}{2} - \frac{j}{2} - \frac{k}{2}\right)(i)\left(\frac{1}{2} + \frac{i}{2} + \frac{j}{2} + \frac{k}{2}\right) = k$. Readers can instead use Equations 15.14 and 15.20, respectively.

15.6 FROM A GIVEN QUATERNION TO ROTATION MATRIX

In this section, we would like to derive the rotation matrix from a given quaternion related to a rigid body rotation. For a given unit quaternion, we can use Equations 15.14 in conjunction with Equation 15.15 for active rotation and Equation 15.20 for passive rotation for calculating the equivalent rotation matrices. However, equivalently we can directly calculate the rotation matrix by implementing quaternions algebraic product rules on $Q\vec{V}Q^*$, for example, for a given vector $\vec{V} = iV_1 + jV_2 + kV_3$.

For a given unit quaternion, we have $Q = Q_0 + \vec{Q} = Q_0 + iQ_1 + jQ_2 + kQ_3$ along with $|Q| = Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2 = 1$. Therefore, the combination $Q\vec{V}Q^*$ can be written as follow.

$Q\vec{V}Q^* = (Q_0 + iQ_1 + jQ_2 + kQ_3)(iV_1 + jV_2 + kV_3)(Q_0 - iQ_1 - jQ_2 - kQ_3)$. Performing the multiplication on the first two brackets, produces $Q\vec{V}Q^* = \left[-(Q_1V_1 + Q_2V_2 + Q_3V_3) + i(Q_0V_1 + Q_2V_3 - Q_3V_2) + j(Q_0V_2 - Q_1V_3 + Q_3V_1) + k(Q_0V_3 + Q_1V_2 - Q_2V_1)\right](Q_0 - iQ_1 - jQ_2 - kQ_3)$. Performing the multiplication on the last bracket produces

$Q\vec{V}Q^* = i\left[V_1(Q_0^2 + Q_1^2 - Q_2^2 - Q_3^2) + 2V_2(Q_1Q_2 - Q_0Q_3) + 2V_3(Q_0Q_2 + Q_1Q_3)\right] + j\left[2V_1(Q_0Q_3 + Q_1Q_2) + V_2(Q_0^2 - Q_1^2 + Q_2^2 - Q_3^2) + 2V_3(Q_2Q_3 - Q_0Q_1)\right] + k\left[2V_1(Q_1Q_3 - Q_0Q_2) + 2V_2(Q_0Q_1 + Q_2Q_3) + V_3(Q_0^2 - Q_1^2 - Q_2^2 + Q_3^2)\right]$. Now using $Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2 = 1$ we can substitute for $-Q_2^2 - Q_3^2 = Q_0^2 + Q_1^2 - 1$, $-Q_1^2 - Q_3^2 = Q_0^2 + Q_2^2 - 1$, and $-Q_1^2 - Q_2^2 = Q_0^2 + Q_3^2 - 1$, to obtain the rotation matrix as: (1st row is the coefficients of i, 2nd row those of j, and 3rd row those of k)

$$[R_{active}] = 2\begin{bmatrix} Q_0^2 + Q_1^2 - 0.5 & Q_1Q_2 - Q_0Q_3 & Q_0Q_2 + Q_1Q_3 \\ Q_0Q_3 + Q_1Q_2 & Q_0^2 + Q_2^2 - 0.5 & Q_2Q_3 - Q_0Q_1 \\ Q_1Q_3 - Q_0Q_2 & Q_0Q_1 + Q_2Q_3 & Q_0^2 + Q_3^2 - 0.5 \end{bmatrix} \quad 15.21$$

Or

$$\begin{Bmatrix} V_1' \\ V_2' \\ V_3' \end{Bmatrix} = [R_{active}]\begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix}$$

Similarly, we can calculate the passive rotation matrix using $Q^*\vec{V}Q$, to receive

$$\left[R_{passive}\right]=2\begin{bmatrix} Q_0^2+Q_1^2-0.5 & Q_0Q_3+Q_1Q_2 & Q_1Q_3-Q_0Q_2 \\ Q_1Q_2-Q_0Q_3 & Q_0^2+Q_2^2-0.5 & Q_0Q_1+Q_2Q_3 \\ Q_0Q_2+Q_1Q_3 & Q_2Q_3-Q_0Q_1 & Q_0^2+Q_3^2-0.5 \end{bmatrix} \qquad 15.22$$

Or

$$\begin{Bmatrix} V_{x'} \\ V_{y'} \\ V_{z'} \end{Bmatrix}=\left[R_{passive}\right]\begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix}$$

Reader may note that $\left[R_{passive}\right]=\left[R_{active}\right]^T$, as expected. Readers should also note that elements of the above-mentioned matrices are calculated using related unit quaternions.

Example: Calculating rotation matrix from a given quaternion-numerical example

The quaternion $q=3+i-2j+k$ is given. Find the equivalent rotation matrix for both active and passive rotation scenarios. For each scenario calculate the equivalent single axis and angle of rotation and compare the results.

Solution:

We first check if the given quaternion is a unit quaternion by calculating its norm/magnitude. Hence for $q=3+i-2j+k$ we have $|q|=\sqrt{9+1+4+1}=\sqrt{15}\neq 1$. Therefore, we need to calculate the corresponding unit quaternion $Q=\frac{1}{\sqrt{15}}(3+i-2j+k)=\left(\frac{3}{\sqrt{15}},\frac{1}{\sqrt{15}},\frac{-2}{\sqrt{15}},\frac{1}{\sqrt{15}}\right)$.

Now using Equation 15.21 for active rotation and plugging in for $Q_0=\frac{3}{\sqrt{15}}$, $Q_1=\frac{1}{\sqrt{15}}$, $Q_2=\frac{-2}{\sqrt{15}}$, and $Q_3=\frac{1}{\sqrt{15}}$ we receive

$$[R_{active}] = 2\begin{bmatrix} \frac{9}{15}+\frac{1}{15}-\frac{1}{2} & -\frac{2}{15}-\frac{3}{15} & -\frac{6}{15}+\frac{1}{15} \\ \frac{3}{15}-\frac{2}{15} & \frac{9}{15}+\frac{4}{15}-\frac{1}{2} & -\frac{2}{15}-\frac{3}{15} \\ \frac{1}{15}+\frac{6}{15} & \frac{3}{15}-\frac{2}{15} & \frac{9}{15}+\frac{1}{15}-\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ \frac{2}{15} & \frac{11}{15} & -\frac{2}{3} \\ \frac{14}{15} & \frac{2}{15} & \frac{1}{3} \end{bmatrix}.$$

The single axis of rotation is the vector corresponding to the vector part of the unit quaternion, or $\vec{Q} = \left(\frac{1}{\sqrt{15}}, \frac{-2}{\sqrt{15}}, \frac{1}{\sqrt{15}}\right)$ and the angle is $2\tan^{-1}\left(\frac{|\vec{Q}|}{Q_0}\right) =$ $2\tan^{-1}\left(\frac{\sqrt{6}}{3}\right) = 78.46^{\circ}$, see Equation 15.15. Note that the corresponding unit vector is

$$\vec{n}\big|_{active} = \frac{\vec{Q}}{|\vec{Q}|} = \frac{\left(\frac{1}{\sqrt{15}}, \frac{-2}{\sqrt{15}}, \frac{1}{\sqrt{15}}\right)}{\sqrt{\frac{1}{15}+\frac{4}{15}+\frac{1}{15}}} = \left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right).$$

Similarly, for passive rotation, using Equation 15.22, we find $\left[R_{passive}\right] = \begin{bmatrix} \frac{1}{3} & \frac{2}{15} & \frac{14}{15} \\ -\frac{2}{3} & \frac{11}{15} & \frac{2}{15} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$. We use the resulted rotation matrix to calculate the single angle of rotation, or $\cos^{-1}\left(\frac{\frac{21}{15}-1}{2}\right) = 78.46^{\circ}$, see Equation 12.8. The axis of rotation is the vector part of the conjugate unit quaternion $Q^* = \frac{1}{\sqrt{15}}(3 - i + 2j - k) = \left(\frac{3}{\sqrt{15}}, \frac{-1}{\sqrt{15}}, \frac{2}{\sqrt{15}}, \frac{-1}{\sqrt{15}}\right)$, or

$\vec{Q}^* = \left(\frac{-1}{\sqrt{15}}, \frac{2}{\sqrt{15}}, \frac{-1}{\sqrt{15}}\right)$ with the unit vector $\vec{n}\big|_{passive} = \dfrac{\left(\frac{-1}{\sqrt{15}}, \frac{2}{\sqrt{15}}, \frac{-1}{\sqrt{15}}\right)}{\sqrt{\frac{1}{15}+\frac{4}{15}+\frac{1}{15}}} =$
$\left(\frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right) = \vec{n}\big|_{active}$.

15.7 FROM A GIVEN ROTATION MATRIX TO QUATERNION

In this section, we would like to derive the equivalent quaternion from a given rotation matrix related to a rigid body rotation. Having the rotation matrix, we can use the relations obtained in the previous section to obtain the corresponding quaternion. Let us consider a passive rotation case, given by Equation 15.22. Note that the trace for both active and passive rotation matrices are equal in their values. The trace of the rotation matrix is $R_{trace} = 2\left(3Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2 - 1.5\right) = 2\left(2Q_0^2 + \underbrace{Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2}_{1} - 1.5\right) = 4Q_0^2 - 1.$ Therefore, we have the real part of the quaternion as

$$Q_0 = \pm\frac{\sqrt{1+R_{trace}}}{2} \qquad 15.23$$

Knowing the real part, we can calculate the elements of complex part $\vec{Q}$ using the diagonal elements of rotation matrix. Therefore, $R(1,1) = 2Q_0^2 + 2Q_1^2 - 1$ or $Q_1 = \pm\sqrt{\frac{R(1,1) - 2Q_0^2 + 1}{2}} = \frac{\sqrt{2R(1,1) - R_{trace} + 1}}{2}$.

Similarly, $q_2 = \pm\frac{\sqrt{2R(2,2) - R_{trace} + 1}}{2}$, and $q_3 = \pm\frac{\sqrt{2R(3,3) - R_{trace} + 1}}{2}$.
Therefore, component m of the complex part of the quaternion reads

$$Q_m = \pm\frac{\sqrt{2R(m,m) - R_{trace} + 1}}{2}, \quad m = 1,2,3 \qquad 15.24$$

From Equations 15.23 and 15.24, it is shown that for a given rotation matrix we get two unit quaternions that describes the rotation, or $Q = Q_0 + Q_1 + Q_2 + Q_3$ and $P = -Q = -Q_0 - Q_1 - Q_2 - Q_3$. Here, Q represents the unit quaternion for a positive (R.H.R.) rotation about the axis of rotation and P represents the rotation about the same axis but in opposite direction, [27].

Example: Calculating quaternion from a given rotation matrix-numerical example

The rotation matrix $R = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is given. Find the corresponding quaternions.

Solution:

Using Equation 15.23, we receive the real part of the corresponding unit quaternion as $Q_0 = \pm\frac{\sqrt{1+0}}{2} = \pm\frac{1}{2}$. The vector part can be calculated using Equation 15.24, or $Q_1 = Q_2 = Q_3 = \pm\frac{\sqrt{0-0+1}}{2} = \pm\frac{1}{2}$. Therefore the quaternions are $Q = \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ and $Q = -\hat{q} = \left(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$.

15.8 FROM EULER ANGLES TO A QUATERNION

In this section, we would like to derive the quaternion from given Euler angles related to a rigid body rotation. Using the relations derived in the previous section, we can find corresponding quaternions for Euler angles. For example, for a roll rotation by angle θ_{roll} about x-axis, we have $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_{roll}) & \sin(\theta_{roll}) \\ 0 & -\sin(\theta_{roll}) & \cos(\theta_{roll}) \end{bmatrix}$, see Table 15.2, as the rotation matrix. Therefore, using Equation 15.23, $|Q_0| = \frac{\sqrt{1+1+2\cos(\theta_{roll})}}{2} = \cos\frac{\theta_{roll}}{2}$, $|Q_1| = \frac{\sqrt{2-2\cos(\theta_{roll})-1+1}}{2} = \sin\frac{\theta_{roll}}{2}$, and $|Q_2| = |Q_3| = \frac{\sqrt{2\cos\theta_{roll} - 2\cos(\theta_{roll})-1+1}}{2} = 0$. This offers the corresponding unit quaternion equivalent to the roll rotation as

$$Q_{\theta_{roll}} = \cos\frac{\theta_{roll}}{2} + i\sin\frac{\theta_{roll}}{2} \tag{15.25}$$

Similarly, the unit quaternions for pitch and yaw rotations are

$$Q_{\theta_{pitch}} = \cos\frac{\theta_{pitch}}{2} + j\sin\frac{\theta_{pitch}}{2} \tag{15.26}$$

and

$$Q_{\theta_{yaw}} = \cos\frac{\theta_{yaw}}{2} + k\sin\frac{\theta_{yaw}}{2} \tag{15.27}$$

Similarly, we can obtain 12 quaternions related to 12 Euler angles combinations. This can be achieved by equating the elements of each rotation matrix given in Table 15.2 and Table 15.3 with Equation 15.22. The result would be nine equations, which can be solved for corresponding quaternion real and imaginary parts [27]. However, in practice it is more efficient to calculate the final rotation matrix and extract the corresponding quaternions, as described in Section 15.7.

Another method of extracting the quaternion related to a given set of Euler angles (without calculating the single-axis rotation matrix) is to use Equation 15.17 in conjunction with Equations 15.25 through 15.27.

15.9 PUTTING IT ALL TOGETHER

It seems useful to collect all related methods and their interrelations, see Table 15.5. In this table, we list the related equations for calculating parameters of a desired method from those of a given method, directly. For example, S→E indicates the direct relations for obtaining Euler angles from a given rotation matrix. Readers should note that it is always possible to calculate the rotation matrix using one of the three methods (i.e., Euler's angle, quaternions, single axis-angle) and then use it for calculating the parameters related to another method.

TABLE 15.5 Rigid body rotation direct relations and methods.

S-single axis-angle	**procedure**	**E-Euler angles**	**procedure**	**Q-quaternions**	**procedure**	**R-rotation matrix**	**procedure**
S→E	S→R→E	E→S	E→R→S	Q→S	Eq. 15.15	R→S	Eq. 12.8, 15.3
S→R	Eq. 12.7	E→R	Table 15- & Table 15-3.	Q→R	Eq. 15.21 &15.22	R→E	Table 15- & Table 15-3.
S→Q	S→R→Q	E→Q	E→R→Q	Q→E	Q→R→E	R→Q	Eq. 15.23, 15.24 & 15.24

CHAPTER 16

WORKED-OUT EXAMPLES

In this section, we present several worked-out examples related to the topics covered in the previous sections.

16.1 EXAMPLE: EINSTEIN SUMMATION CONVENTIONS

Write out the expanded expression in full detail for $N = 3$ dimensions for, $A = dx^i \left(\vec{e}_i - \frac{\partial x'^j}{\partial x^i} \vec{e'}_j \right)$.

Solution:

Summation applies on both indices i and j, hence $A = dx^i \left(\vec{e}_i - \frac{\partial x'^j}{\partial x^i} \vec{e'}_j \right) =$

$$dx^i \vec{e}_i - dx^i \frac{\partial x'^j}{\partial x^i} \vec{e'}_j = dx^1 \vec{e}_1 + dx^2 \vec{e}_2 + dx^3 \vec{e}_3 - dx^1 \left(\frac{\partial x'^1}{\partial x^1} \vec{e'_1} + \frac{\partial x'^2}{\partial x^1} \vec{e'_2} + \frac{\partial x'^3}{\partial x^1} \vec{e'_3} \right) -$$

$$dx^2 \left(\frac{\partial x'^1}{\partial x^2} \vec{e'_1} + \frac{\partial x'^2}{\partial x^2} \vec{e'_2} + \frac{\partial x'^3}{\partial x^2} \vec{e'_3} \right) - dx^3 \left(\frac{\partial x'^1}{\partial x^3} \vec{e'_1} + \frac{\partial x'^2}{\partial x^3} \vec{e'_2} + \frac{\partial x'^3}{\partial x^3} \vec{e'_3} \right).$$

Re-arranging the terms, we receive $A = dx^1 \left(\vec{e}_1 - \frac{\partial x'^1}{\partial x^1} \vec{e'_1} - \frac{\partial x'^2}{\partial x^1} \vec{e'_2} - \frac{\partial x'^3}{\partial x^1} \vec{e'_3} \right) +$

$$dx^2 \left(\vec{e}_2 - \frac{\partial x'^1}{\partial x^2} \vec{e'_1} - \frac{\partial x'^2}{\partial x^2} \vec{e'_2} - \frac{\partial x'^3}{\partial x^2} \vec{e'_3} \right) + dx^3 \left(\vec{e}_3 - \frac{\partial x'^1}{\partial x^3} \vec{e'_1} - \frac{\partial x'^2}{\partial x^3} \vec{e'_2} - \frac{\partial x'^3}{\partial x^3} \vec{e'_3} \right).$$

16.2 EXAMPLE: CONVERSION FROM VECTOR TO INDEX NOTATIONS

Write out the following expressions written in vector form in index notation and expand for 3D coordinates.

a. $\left(\vec{\nabla}\cdot\vec{A}\right)^2$

b. $\vec{A}\cdot\left(\nabla^2\vec{A}\right)$

c. $\vec{A}\cdot\left(\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)\right)$

d. $\left(\vec{\nabla}\cdot\vec{A}\right)\left(\nabla^2\Phi\right)$

e. $\nabla^2\left(\vec{A}\cdot(\nabla\Phi)\right)$

f. $\vec{A}\cdot\left(\vec{\nabla}\left(\nabla^2\Phi\right)\right)$

Solution:

a. $\left(\vec{\nabla}\cdot\vec{A}\right)^2 = A^j_{,j}A^k_{,k} = \left(A^1_{,1}+A^2_{,2}+A^3_{,3}\right)^2$

b. $\vec{A}\cdot\left(\nabla^2\vec{A}\right) = A^k A_{k}{}_{,\cdot j}^{\;\;j} = A^1 A_{1}{}_{,\cdot j}^{\;\;j} + A^2 A_{2}{}_{,\cdot j}^{\;\;j} + A^3 A_{3}{}_{,\cdot j}^{\;\;j} = A^1\left(A_{1}{}_{,\cdot 1}^{\;\;1} + A_{1}{}_{,\cdot 2}^{\;\;2} + A_{1}{}_{,\cdot 3}^{\;\;3}\right)$

$$+A^2\left(A_{1}{}_{,\cdot 1}^{\;\;1} + A_{1}{}_{,\cdot 2}^{\;\;2} + A_{1}{}_{,\cdot 3}^{\;\;3}\right) + A^3\left(A_{1}{}_{,\cdot 1}^{\;\;1} + A_{1}{}_{,\cdot 2}^{\;\;2} + A_{1}{}_{,\cdot 3}^{\;\;3}\right)$$

c. $\vec{A}\cdot\left(\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)\right) = A^j A^k_{,kj} = A^1A^k_{,k1} + A^2A^k_{,k2} + A^3A^k_{,k3} = A^1\left(A^1_{,11}+A^2_{,21}+A^3_{,31}\right)$

$$+A^2\left(A^1_{,11}+A^2_{,21}+A^3_{,31}\right) + A^3\left(A^1_{,11}+A^2_{,21}+A^3_{,31}\right)$$

d. $\left(\vec{\nabla}\cdot\vec{A}\right)\left(\nabla^2\Phi\right) = A^j_{,j}\left(\nabla_k\Phi\right)_{,\cdot}^{\;k} = A^1_{,1}\left(\nabla_k\Phi\right)_{,\cdot}^{\;k} + A^2_{,2}\left(\nabla_k\Phi\right)_{,\cdot}^{\;k} + A^3_{,3}\left(\nabla_k\Phi\right)_{,\cdot}^{\;k} =$

$$A^1_{,1}\left[\left(\nabla_1\Phi\right)_{,\cdot}^{\;1} + \left(\nabla_2\Phi\right)_{,\cdot}^{\;2} + \left(\nabla_3\Phi\right)_{,\cdot}^{\;3}\right] + A^2_{,2}\left[\left(\nabla_1\Phi\right)_{,\cdot}^{\;1} + \left(\nabla_2\Phi\right)_{,\cdot}^{\;2} + \left(\nabla_3\Phi\right)_{,\cdot}^{\;3}\right] +$$

$$A^3_{,3}\left[\left(\nabla_1\Phi\right)_{,\cdot}^{\;1} + \left(\nabla_2\Phi\right)_{,\cdot}^{\;2} + \left(\nabla_3\Phi\right)_{,\cdot}^{\;3}\right]$$

e. $$\nabla^2\left(\vec{A}\cdot(\nabla\Phi)\right)=\left(A^k(\nabla_k\Phi)\right)_{,j\cdot}{}^{j}$$
$$=\left(A^1(\nabla_1\Phi)+A^2(\nabla_2\Phi)+A^3(\nabla_3\Phi)\right)_{,j\cdot}{}^{j}$$
$$=\left(A^1(\nabla_1\Phi)+A^2(\nabla_2\Phi)+A^3(\nabla_3\Phi)\right)_{,1\cdot}{}^{1}$$
$$+\left(A^1(\nabla_1\Phi)+A^2(\nabla_2\Phi)+A^3(\nabla_3\Phi)\right)_{,2\cdot}{}^{2}$$
$$+\left(A^1(\nabla_1\Phi)+A^2(\nabla_2\Phi)+A^3(\nabla_3\Phi)\right)_{,3\cdot}{}^{3}$$

f. $$\vec{A}\cdot\left(\vec{\nabla}\left(\nabla^2\Phi\right)\right)=A^j(\nabla_k\Phi)_{,\cdot j}^{\ k}$$
$$=A^1(\nabla_k\Phi)_{,\cdot 1}^{\ k}+A^2(\nabla_k\Phi)_{,\cdot 2}^{\ k}+A^3(\nabla_k\Phi)_{,\cdot 3}^{\ k}$$
$$=A^1\left[(\nabla_1\Phi)_{,\cdot 1}^{\ 1}+(\nabla_2\Phi)_{,\cdot 1}^{\ 2}+(\nabla_3\Phi)_{,\cdot 1}^{\ 3}\right]$$
$$+A^2\left[(\nabla_1\Phi)_{,\cdot 2}^{\ 1}+(\nabla_2\Phi)_{,\cdot 2}^{\ 2}+(\nabla_3\Phi)_{,\cdot 2}^{\ 3}\right]$$
$$+A^3\left[(\nabla_1\Phi)_{,\cdot 3}^{\ 1}+(\nabla_2\Phi)_{,\cdot 3}^{\ 2}+(\nabla_3\Phi)_{,\cdot 3}^{\ 3}\right]$$

16.3 EXAMPLE: OBLIQUE RECTILINEAR COORDINATE SYSTEMS

Consider a 2D oblique/slanted coordinate system, x^i in which the coordinate axes are not at right angles. With reference to the Cartesian coordinate system y^i, as shown in Figure 16.1, for a given vector $\vec{A}$ find:

a. Covariant and contravariant components; A_i and A^i

b. Covariant and contravariant basis vectors; $\vec{e}_i$ and $\vec{e}^{\,i}$ with sketch on the coordinate system

c. Show that $\vec{A}=A^i\vec{e}_i=A_i\vec{e}^{\,i}=A_{c1}\vec{E}_1+A_{c2}\vec{E}_2$; where $\vec{E}_1$ and $\vec{E}_2$ are Cartesian unit vectors

d. Verify that $\vec{e}^{\,i}\cdot\vec{e}_j=\delta^i_j$ and sketch all basis vectors

e. Metric tensors

f. Unit vectors $\vec{e}_i/\left|\vec{e}_i\right|$, or scale factor, and $\vec{e}^{\,i}/\left|\vec{e}^{\,i}\right|$

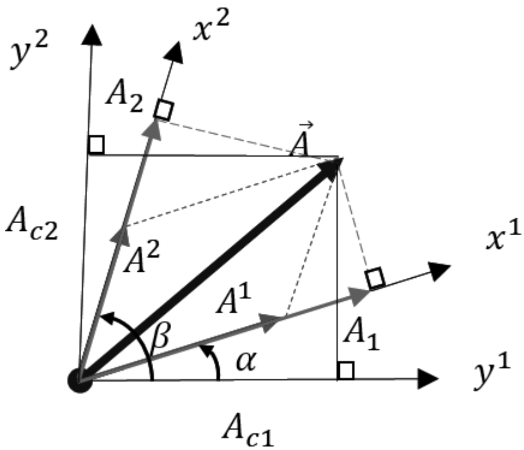


FIGURE 16.1 Oblique coordinate system x^i and Cartesian coordinate system y^i.

Solution:

Assume axes x^1 and x^2 make angles α and β with Cartesian axis y^1, respectively. Therefore, we can write the following functional relations between the oblique and Cartesian systems:

$$\begin{cases} y^1 = x^1\cos\alpha + x^2\cos\beta \\ y^2 = x^1\sin\alpha + x^2\sin\beta \end{cases}, \text{ or inversely } \begin{cases} x^1 = -\csc(\alpha-\beta)\left[y^1\sin\beta - y^2\cos\beta\right] \\ x^2 = \csc(\alpha-\beta)\left[y^1\sin\alpha - y^2\cos\alpha\right] \end{cases}$$

a. Covariant components of $\vec{A}$ are obtained by drawing perpendicular lines to the coordinate axes. We have $A_i = \frac{\partial y^j}{\partial x^i}A_{cj}$, or $A_1 = \frac{\partial y^1}{\partial x^1}A_{c1} + \frac{\partial y^2}{\partial x^1}A_{c2} = \cos\alpha A_{c1} + \sin\alpha A_{c2}$. Similarly, $A_2 = \frac{\partial y^1}{\partial x^2}A_{c1} + \frac{\partial y^2}{\partial x^2}A_{c2} = \cos\beta A_{c1} + \sin\beta A_{c2}$. The contravariant components of $\vec{A}$ are obtained by drawing parallel lines to the coordinate axes. We have $A^i = \frac{\partial x^i}{\partial y^j}A_{cj}$, or $A^1 = \frac{\partial x^1}{\partial y^1}A_{c1} + \frac{\partial x^1}{\partial y^2}A_{c2} = -\sin\beta\csc(\alpha-\beta)A_{c1} + \cos\beta\csc(\alpha-\beta)A_{c2}$. Similarly, $A^2 = \frac{\partial x^2}{\partial y^1}A_{c1} + \frac{\partial x^2}{\partial y^2}A_{c2} = \sin\alpha\csc(\alpha-\beta)A_{c1} - \cos\alpha\csc(\alpha-\beta)A_{c2}$. Note that for the Cartesian system both covariant and contravariant components are identical.

b. The covariant basis vectors for oblique system are $\vec{e}_i = \frac{\partial y^j}{\partial x^i}\vec{E}_j$, or $\vec{e}_1 = \frac{\partial y^1}{\partial x^1}\vec{E}_1 + \frac{\partial y^2}{\partial x^1}\vec{E}_2 = \cos\alpha\,\vec{E}_1 + \sin\alpha\,\vec{E}_2$. Similarly, $\vec{e}_2 = \frac{\partial y^1}{\partial x^2}\vec{E}_1 + \frac{\partial y^2}{\partial x^2}\vec{E}_2 = \cos\beta\,\vec{E}_1 + \sin\beta\,\vec{E}_2$. The contravariant basis vectors are $\vec{e}^{\,i} = \frac{\partial x^i}{\partial y^j}\vec{E}_j$,

or $\vec{e}^{\,1} = \frac{\partial x^1}{\partial y^1}\vec{E}_1 + \frac{\partial x^1}{\partial y^2}\vec{E}_2 = -\sin\beta\csc(\alpha-\beta)\vec{E}_1 + \cos\beta\csc(\alpha-\beta)\vec{E}_2$.

Similarly, $\vec{e}^{\,2} = \frac{\partial x^2}{\partial y^1}\vec{E}_1 + \frac{\partial x^2}{\partial y^2}\vec{E}_2 = \sin\alpha\csc(\alpha-\beta)\vec{E}_1 - \cos\alpha\csc(\alpha-\beta)\vec{E}_2$.

Note that for the Cartesian system both covariant and contravariant basis vectors are identical and are unit vectors, i.e. $\left|\vec{E}_i\right| = 1$.

c. $A^i\vec{e}_i = A^1\vec{e}_1 + A^2\vec{e}_2 = [-\sin\beta\csc(\alpha-\beta)A_{c1} + \cos\beta\csc(\alpha-\beta)A_{c2}][\cos\alpha\,\vec{E}_1 + \sin\alpha\,\vec{E}_2] + [\sin\alpha\csc(\alpha-\beta)A_{c1} - \cos\alpha\csc(\alpha-\beta)A_{c2}][\cos\beta\,\vec{E}_1 + \sin\beta\,\vec{E}_2]$. After some manipulations and rearranging the terms, we find $A^i\vec{e}_i = A_{c1}\,\vec{E}_1 + A_{c2}\,\vec{E}_2$. Similarly, $A_i\vec{e}^{\,i} = A_1\vec{e}^{\,1} + A_2\vec{e}^{\,2} = [\cos\alpha A_{c1} + \sin\alpha A_{c2}][-\sin\beta\csc(\alpha-\beta)\vec{E}_1 + \cos\beta\csc(\alpha-\beta)\vec{E}_2] + [\cos\beta A_{c1} + \sin\beta A_{c2}][\sin\alpha\csc(\alpha-\beta)\vec{E}_1 - \cos\alpha\csc(\alpha-\beta)\vec{E}_2]$. After some manipulations and rearranging the terms, we find $A_i\vec{e}^{\,i} = A_{c1}\,\vec{E}_1 + A_{c2}\,\vec{E}_2$.

d. Dot-product of covariant and contravariant basis vectors can now be written as $\vec{e}_1\cdot\vec{e}^{\,2} = \vec{e}^{\,2}\cdot\vec{e}_1 = \frac{\cos\alpha\sin\alpha}{\sin(\alpha-\beta)} - \frac{\sin\alpha\cos\alpha}{\sin(\alpha-\beta)} = 0$ and $\vec{e}_2\cdot\vec{e}^{\,1} = \vec{e}^{\,1}\cdot\vec{e}_2 = -\frac{\cos\beta\sin\beta}{\sin(\alpha-\beta)} + \frac{\sin\beta\cos\beta}{\sin(\alpha-\beta)} = 0$. These results confirm that $\vec{e}_1 \perp \vec{e}^{\,2}$ and $\vec{e}_2 \perp \vec{e}^{\,1}$, as shown in Figure 16.2. Also, we can write $\vec{e}_1\cdot\vec{e}^{\,1} = -\frac{\cos\alpha\sin\beta}{\sin(\alpha-\beta)} + \frac{\sin\alpha\cos\beta}{\sin(\alpha-\beta)} = 1$ and $\vec{e}_2\cdot\vec{e}^{\,2} = \frac{\cos\beta\sin\alpha}{\sin(\alpha-\beta)} - \frac{\sin\beta\cos\alpha}{\sin(\alpha-\beta)} = 1$. These results verify that $\vec{e}^{\,i}\cdot\vec{e}_j = \delta^i_j$.

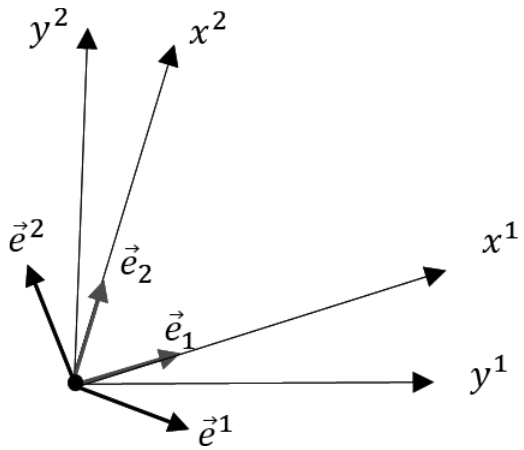


FIGURE 16.2 Covariant and contravariant basis vectors in oblique coordinate systems.

e. The doubly covariant metric tensor $g_{ij} = \begin{bmatrix} \vec{e}_1 \cdot \vec{e}_1 & \vec{e}_1 \cdot \vec{e}_2 \\ \vec{e}_2 \cdot \vec{e}_1 & \vec{e}_2 \cdot \vec{e}_2 \end{bmatrix} = \begin{bmatrix} 1 & \cos(\alpha-\beta) \\ \cos(\alpha-\beta) & 1 \end{bmatrix}$. We can calculate doubly contravariant metric tensor by inverting the g_{ij}, or using $g^{ij} = \begin{bmatrix} \vec{e}^1 \cdot \vec{e}^1 & \vec{e}^1 \cdot \vec{e}^2 \\ \vec{e}^2 \cdot \vec{e}^1 & \vec{e}^2 \cdot \vec{e}^2 \end{bmatrix} = \csc^2(\alpha-\beta) \begin{bmatrix} 1 & -\cos(\alpha-\beta) \\ -\cos(\alpha-\beta) & 1 \end{bmatrix}$. These results confirm that x^i is a non-orthogonal system, since off-diagonal elements of metric tensors are not equal to zero for all values of angles α and β with the condition that $(\alpha-\beta) \neq (2k+1)\frac{\pi}{2}$. where, k is an integer.

f. Unit vectors along the covariant basis vectors are; $\frac{\vec{e}_1}{|\vec{e}_1|} = \cos\alpha \vec{E}_1 + \sin\alpha \vec{E}_2$ and $\frac{\vec{e}_2}{|\vec{e}_2|} = \cos\beta \vec{E}_1 + \sin\beta \vec{E}_2$. The unit vectors along the contravariant basis vectors are; $\frac{\vec{e}^1}{|\vec{e}^1|} = \frac{\vec{e}^1}{\sqrt{\vec{e}^1 \cdot \vec{e}^1}} = -\sin\beta \vec{E}_1 + \cos\beta \vec{E}_2$ and $\frac{\vec{e}^2}{|\vec{e}^2|} = \frac{\vec{e}^2}{\sqrt{\vec{e}^2 \cdot \vec{e}^2}} = \sin\alpha \vec{E}_1 - \cos\alpha \vec{E}_2$.

16.4 EXAMPLE: QUANTITIES RELATED TO PARABOLIC COORDINATE SYSTEM

Consider a parabolic coordinate system $\left(x^1, x^2, x^3\right) \equiv (\xi, \eta, \theta)$ given in terms of Cartesian $\left(y^1, y^2, y^3\right) \equiv (X, Y, Z)$, as shown in Figure 16.3. The functional relations are

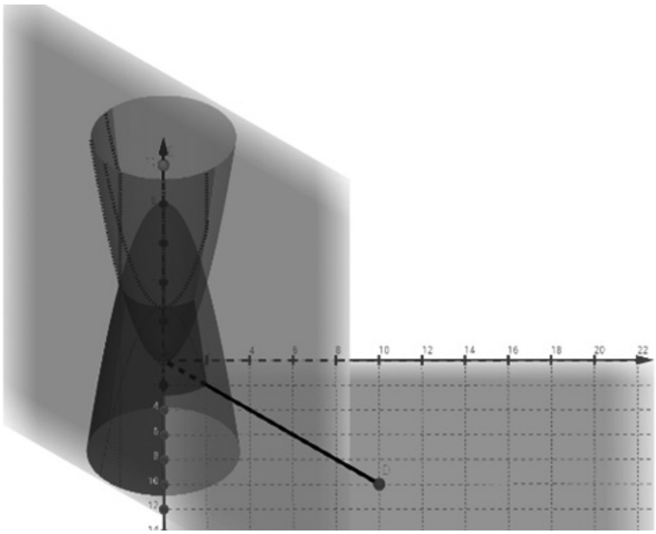


FIGURE 16.3 Parabolic coordinate system.

$$\begin{cases} X = \xi\eta\cos\theta \\ Y = \xi\eta\sin\theta \\ Z = \left(\xi^2 - \eta^2\right)/2 \end{cases}$$

. Find the basis vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ for the parabolic coordinate system in terms of the Cartesian unit vectors. Also find the scale factors, unit vectors, metric tensors (covariant and contravariant), Jacobian, volume element, and Christoffel symbols of the 2nd kind.

Solution:

The covariant basis vector $\left(\vec{e}_\xi, \vec{e}_\eta, \vec{e}_\theta\right)$ reads, using $\vec{e}_k = \dfrac{\partial y^j}{\partial x^k}\vec{E}_j$

$$\begin{cases} \vec{e}_\xi = \eta\cos\theta\vec{E}_1 + \eta\sin\theta\vec{E}_2 + \xi\vec{E}_3 \\ \vec{e}_\eta = \xi\cos\theta\vec{E}_1 + \xi\sin\theta\vec{E}_2 - \eta\vec{E}_3 \\ \vec{e}_\theta = -\xi\eta\sin\theta\vec{E}_1 + \xi\eta\cos\theta\vec{E}_2 \end{cases}$$

The scale factors are $h_\xi = \left|\vec{e}_\xi\right| = \sqrt{\xi^2 + \eta^2}$, $h_\eta = \left|\vec{e}_\eta\right| = \sqrt{\xi^2 + \eta^2}$, and $h_\theta = \xi\eta$. The unit vectors are, using $\vec{e}(k) = \vec{e}_k / h_k$,

$$\begin{cases} \vec{e}(\xi) = \left(\eta\cos\theta\vec{E}_1 + \eta\sin\theta\vec{E}_2 + \xi\vec{E}_3\right) / \sqrt{\xi^2 + \eta^2} \\ \vec{e}(\eta) = \left(\xi\cos\theta\vec{E}_1 + \xi\sin\theta\vec{E}_2 - \eta\vec{E}_3\right) / \sqrt{\xi^2 + \eta^2} \\ \vec{e}(\theta) = -\sin\theta\vec{E}_1 + \cos\theta\vec{E}_2 \end{cases}$$

Covariant metric tensor reads, using $g_{ij} = \vec{e}_i \cdot \vec{e}_j$, $g_{ij} = \begin{bmatrix} \xi^2 + \eta^2 & 0 & 0 \\ 0 & \xi^2 + \eta^2 & 0 \\ 0 & 0 & (\xi\eta)^2 \end{bmatrix}$. Therefore, the parabolic coordinate system is an orthogonal one, since metric tensor is a diagonal matrix with null off-diagonal elements. The contravariant metric tensor is the inverse of the covariant one, or $g^{ij} = \left(g_{ij}\right)^{-1} = \begin{bmatrix} \left(\xi^2 + \eta^2\right)^{-1} & 0 & 0 \\ 0 & \left(\xi^2 + \eta^2\right)^{-1} & 0 \\ 0 & 0 & (\xi\eta)^{-2} \end{bmatrix}$.

Jacobian is the square root of the determinant of g_{ij}, or the product of all scale factors, $\mathcal{J} = \sqrt{\left|g_{ij}\right|} = \xi\eta\left(\xi^2 + \eta^2\right)$. The volume element is $d\mathbb{V} = \mathcal{J} d\xi d\eta d\theta = \xi\eta\left(\xi^2 + \eta^2\right) d\xi d\eta d\theta$. We use Equation 10.20, or $\Gamma_{ij}^m = \frac{g^{mk}}{2}\left[\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right]$ to calculate the Christoffel symbols. Therefore, $\Gamma_{11}^1 = \frac{g^{11}}{2}\left[\frac{\partial g_{11}}{\partial \xi}\right] = \frac{\xi}{\xi^2 + \eta^2}$,

$\Gamma_{12}^1 = \frac{g^{11}}{2}\left[\frac{\partial g_{11}}{\partial \eta}\right] = \frac{\eta}{\xi^2 + \eta^2}$, $\Gamma_{13}^1 = 0$, $\Gamma_{11}^2 = \frac{g^{22}}{2}\left[-\frac{\partial g_{11}}{\partial \eta}\right] = -\frac{\eta}{\xi^2 + \eta^2}$,

$\Gamma_{12}^2 = \frac{g^{22}}{2}\left[\frac{\partial g_{22}}{\partial \xi}\right] = \frac{\xi}{\xi^2 + \eta^2}$, $\Gamma_{13}^2 = \Gamma_{11}^3 = \Gamma_{12}^3 = 0$, $\Gamma_{13}^3 = \frac{g^{33}}{2}\left[\frac{\partial g_{33}}{\partial \xi}\right] = \frac{1}{\xi}$,

$\Gamma^1_{22} = \frac{g^{11}}{2}\left[-\frac{\partial g_{22}}{\partial \xi}\right] = -\frac{\xi}{\xi^2+\eta^2}$, $\Gamma^1_{23} = 0$, $\Gamma^1_{33} = \frac{g^{11}}{2}\left[-\frac{\partial g_{33}}{\partial \xi}\right] = -\frac{\xi\eta^2}{\xi^2+\eta^2}$,

$\Gamma^2_{22} = \frac{g^{22}}{2}\left[\frac{\partial g_{22}}{\partial \eta}\right] = \frac{\eta}{\xi^2+\eta^2}$, $\Gamma^2_{23} = 0$, $\Gamma^2_{33} = \frac{g^{22}}{2}\left[-\frac{\partial g_{33}}{\partial \eta}\right] = -\frac{\eta\xi^2}{\xi^2+\eta^2}$,

$\Gamma^3_{23} = \frac{g^{33}}{2}\left[\frac{\partial g_{33}}{\partial \eta}\right] = \frac{1}{\eta}$, $\Gamma^3_{33} = 0$. In matrix form, we have

$$\Gamma^{\xi} = \begin{bmatrix} \Gamma^{\xi}_{\xi\xi} & \Gamma^{\xi}_{\xi\eta} & \Gamma^{\xi}_{\xi\theta} \\ \Gamma^{\xi}_{\eta\xi} & \Gamma^{\xi}_{\eta\eta} & \Gamma^{\xi}_{\eta\theta} \\ \Gamma^{\xi}_{\theta\xi} & \Gamma^{\xi}_{\theta\eta} & \Gamma^{\xi}_{\theta\theta} \end{bmatrix} = \begin{bmatrix} \frac{\xi}{\xi^2+\eta^2} & \frac{\eta}{\xi^2+\eta^2} & 0 \\ \frac{\eta}{\xi^2+\eta^2} & -\frac{\xi}{\xi^2+\eta^2} & 0 \\ 0 & 0 & -\frac{\xi\eta^2}{\xi^2+\eta^2} \end{bmatrix}$$

$$\Gamma^{\eta} = \begin{bmatrix} \Gamma^{\eta}_{\xi\xi} & \Gamma^{\eta}_{\xi\eta} & \Gamma^{\eta}_{\xi\theta} \\ \Gamma^{\eta}_{\eta\xi} & \Gamma^{\eta}_{\eta\eta} & \Gamma^{\eta}_{\eta\theta} \\ \Gamma^{\eta}_{\theta\xi} & \Gamma^{\eta}_{\theta\eta} & \Gamma^{\eta}_{\theta\theta} \end{bmatrix} = \begin{bmatrix} -\frac{\eta}{\xi^2+\eta^2} & \frac{\xi}{\xi^2+\eta^2} & 0 \\ \frac{\xi}{\xi^2+\eta^2} & \frac{\eta}{\xi^2+\eta^2} & 0 \\ 0 & 0 & -\frac{\eta\xi^2}{\xi^2+\eta^2} \end{bmatrix}$$

$$\Gamma^{\theta} = \begin{bmatrix} \Gamma^{\theta}_{\xi\xi} & \Gamma^{\theta}_{\xi\eta} & \Gamma^{\theta}_{\xi\theta} \\ \Gamma^{\theta}_{\eta\xi} & \Gamma^{\theta}_{\eta\eta} & \Gamma^{\theta}_{\eta\theta} \\ \Gamma^{\theta}_{\theta\xi} & \Gamma^{\theta}_{\theta\eta} & \Gamma^{\theta}_{\theta\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{\xi} \\ 0 & 0 & \frac{1}{\eta} \\ \frac{1}{\xi} & \frac{1}{\eta} & 0 \end{bmatrix}$$

16.5 EXAMPLE: QUANTITIES RELATED TO BI-POLAR COORDINATE SYSTEMS

Consider a bi-polar coordinate system $\left(x^1, x^2, x^3\right) \equiv (\xi, \eta, z)$ given in terms of Cartesian $\left(y^1, y^2, y^3\right) \equiv (X, Y, Z)$, as shown in Figure 16.4. The functional relations are $\begin{cases} X = \dfrac{a \sinh \eta}{\cosh \eta - \cos \xi} \\ Y = \dfrac{a \sin \xi}{\cosh \eta - \cos \xi} \\ Z = z \end{cases}$. Find the basis vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ for the bi-polar coordinate system in terms of the Cartesian unit vectors. Also find the scale factors, unit vectors, metric tensors (covariant and contravariant), Jacobian, volume element, and Christoffel symbols of the 2nd kind.

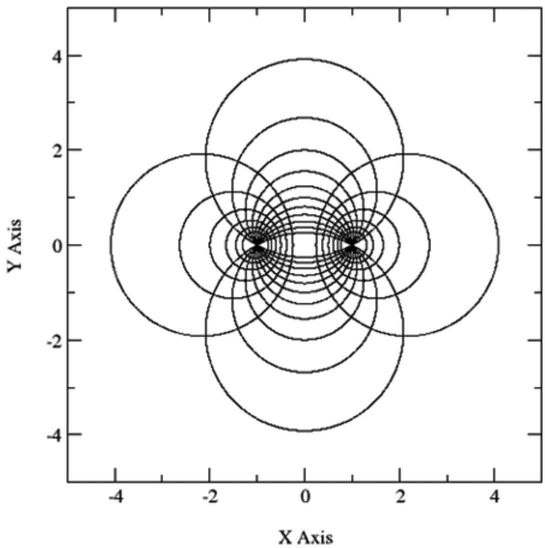


FIGURE 16.4 Bipolar coordinates: ξ and η iso-surfaces (Foci are located at (−1, 0) and (1, 0)[1].

[1] (https://commons.wikimedia.org/wiki/File:Bipolar_isosurfaces.png)

Solution:

The covariant basis vector $\left(\vec{e}_{\xi},\vec{e}_{\eta},\vec{e}_{z}\right)$ reads, using $\vec{e}_k = \dfrac{\partial y^j}{\partial x^k}\vec{E}_j$

$$\begin{cases} \vec{e}_{\xi} = \dfrac{-a\sinh\eta\sin\xi}{\left(\cosh\eta-\cos\xi\right)^2}\vec{E}_1 + \dfrac{a\left(\cosh\eta\cos\xi-1\right)}{\left(\cosh\eta-\cos\xi\right)^2}\vec{E}_2 \\ \vec{e}_{\eta} = \dfrac{a\left(1-\cosh\eta\cos\xi\right)}{\left(\cosh\eta-\cos\xi\right)^2}\vec{E}_1 - \dfrac{a\sinh\eta\sin\xi}{\left(\cosh\eta-\cos\xi\right)^2}\vec{E}_2 \\ \vec{e}_z = \vec{E}_3 \end{cases}$$

The scale factors, after simplification, are $h_{\xi} = \left|\vec{e}_{\xi}\right| = \dfrac{a}{\cosh\eta-\cos\xi}$, $h_{\eta} = \left|\vec{e}_{\eta}\right| = \dfrac{a}{\cosh\eta-\cos\xi}$, and $h_z = 1$. The unit vectors are, using $\vec{e}\left(k\right) = \vec{e}_k / h_k$,

$$\begin{cases} \vec{e}\left(\xi\right) = \dfrac{-\sinh\eta\sin\xi}{\cosh\eta-\cos\xi}\vec{E}_1 + \dfrac{\cosh\eta\cos\xi-1}{\cosh\eta-\cos\xi}\vec{E}_2 \\ \vec{e}\left(\eta\right) = \dfrac{1-\cosh\eta\cos\xi}{\cosh\eta-\cos\xi}\vec{E}_1 - \dfrac{\sinh\eta\sin\xi}{\cosh\eta-\cos\xi}\vec{E}_2 \\ \vec{e}\left(z\right) = \vec{E}_3 \end{cases}$$

The covariant metric tensor reads, using $g_{ij} = \vec{e}_i \cdot \vec{e}_j$, $g_{ij} = \begin{bmatrix} \dfrac{a^2}{\left(\cosh\eta-\cos\xi\right)^2} & 0 & 0 \\ 0 & \dfrac{a^2}{\left(\cosh\eta-\cos\xi\right)^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Therefore, the parabolic coordinate system is an orthogonal one, since metric tensor is a diagonal matrix with null off-diagonal elements. The contravariant metric tensor is the inverse of the covariant one, or $g^{ij} = \left(g_{ij}\right)^{-1} = \begin{bmatrix} \frac{(\cosh\eta - \cos\xi)^2}{a^2} & 0 & 0 \\ 0 & \frac{(\cosh\eta - \cos\xi)^2}{a^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Jacobian is the square root of the determinant of g_{ij}, or the product of all scale factors, $\mathcal{J} = \sqrt{\left|g_{ij}\right|} = \frac{a^2}{(\cosh\eta - \cos\xi)^2}$. The volume element is $d\mathbb{V} = \mathcal{J} d\xi d\eta d\theta = \frac{a^2}{(\cosh\eta - \cos\xi)^2} d\xi d\eta dz$. Similarly, as given in Example 16.5, the Christoffel symbols can be calculated as

$$\Gamma^{\xi} = \begin{bmatrix} \Gamma^{\xi}_{\xi\xi} & \Gamma^{\xi}_{\xi\eta} & \Gamma^{\xi}_{\xi z} \\ \Gamma^{\xi}_{\eta\xi} & \Gamma^{\xi}_{\eta\eta} & \Gamma^{\xi}_{\eta z} \\ \Gamma^{\xi}_{z\xi} & \Gamma^{\xi}_{z\eta} & \Gamma^{\xi}_{zz} \end{bmatrix} = \begin{bmatrix} \frac{\sin\xi}{\cos\xi - \cosh\eta} & \frac{\sinh\eta}{\cos\xi - \cosh\eta} & 0 \\ \frac{\sinh\eta}{\cos\xi - \cosh\eta} & \frac{-\sin\xi}{\cos\xi - \cosh\eta} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma^{\eta} = \begin{bmatrix} \Gamma^{\eta}_{\xi\xi} & \Gamma^{\eta}_{\xi\eta} & \Gamma^{\eta}_{\xi z} \\ \Gamma^{\eta}_{\eta\xi} & \Gamma^{\eta}_{\eta\eta} & \Gamma^{\eta}_{\eta z} \\ \Gamma^{\eta}_{z\xi} & \Gamma^{\eta}_{z\eta} & \Gamma^{\eta}_{zz} \end{bmatrix} = \begin{bmatrix} \frac{-\sinh\eta}{\cos\xi - \cosh\eta} & \frac{\sin\xi}{\cos\xi - \cosh\eta} & 0 \\ \frac{\sin\xi}{\cos\xi - \cosh\eta} & \frac{\sinh\eta}{\cos\xi - \cosh\eta} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma^{z} = [0]$$

16.6 EXAMPLE: APPLICATION OF CONTRAVARIANT METRIC TENSORS

Using the property of contravariant metric tensors, show that the following relation holds, $g^{il}\varepsilon_{ljk}A^jB^k = \varepsilon^{ijk}A_jB_k$.

Solution:

Knowing that metric tensors replace repeated indices with the free index while raising (contravariant metric tensor) or lowering (covariant metric tensor) them. We can write the expression in the L.H.S of the above relation as, $\underbrace{g^{il}\varepsilon_{ljk}}A^jB^k = \mathcal{E}^i_{jk}A^jB^k$. But we can write $A^j = g^{jn}A_n$ and $B^k = g^{km}B_m$, substituting, gives $\underbrace{g^{il}\varepsilon_{ljk}}A^jB^k = \mathcal{E}^i_{jk}g^{jn}A_ng^{km}B_m = \underbrace{\mathcal{E}^i_{jk}g^{jn}}g^{km}A_nB_m = \underbrace{\mathcal{E}^{in}_{k}g^{km}}A_nB_m = \mathcal{E}^{inm}A_nB_m$, after changing dummy indices, i.e., $n \to j$ and $m \to k$, we get $\mathcal{E}^{inm}A_nB_m = \mathcal{E}^{ijk}A_jB_k$. Therefore, the given relation holds.

16.7 EXAMPLE: DOT AND CROSS PRODUCTS IN CYLINDRICAL AND SPHERICAL COORDINATES

Given that $A(i)$ and $B(i)$ are the physical components of vectors $\vec{A}$ and $\vec{B}$, respectively, write down the expressions for physical components of the cross-product, $\vec{A}\times\vec{B}$, and dot-product $\vec{A}\cdot\vec{B}$ in terms of $A(i)$ and $B(i)$, for the following coordinate systems (assume a 3D space, $N = 3$):

a. Cylindrical polar $\left(x^1, x^2, x^3\right) \equiv (r, \theta, z)$

b. Spherical polar $\left(x^1, x^2, x^3\right) \equiv (r, \varphi, \theta)$

Solution:

The dot-product and cross-product relations are $\vec{A}\cdot\vec{B} = A^iB_i$, $C^i = \mathcal{E}^{ijk}A_jB_k$ (contravariant component). We use the relations between covariant/contravariant and physical components, or $A_i = h_iA(i)$ and $A^i = A(i)/h_i$.

a. For cylindrical coordinates, we have the scale factors as $(h_1, h_2, h_3) = (1, r, 1)$ and the Jacobian is $\mathcal{J} = r$ (see Section 11.6.1). We can write $B_1 = h_1B(1) = B(1)$, $B_2 = h_2B(2) = rA(2)$, $B_3 = h_3B(3) = B(3)$.

Similarly, $A^1 = \dfrac{A(1)}{h_1} = A(1)$, $A^2 = \dfrac{A(2)}{h_2} = \dfrac{A(2)}{r}$, $A^3 = \dfrac{A(3)}{h_3} = A(3)$. Therefore, the dot-product reads

$\vec{A} \cdot \vec{B} = A^i B_i = A^1 B_1 + A^2 B_2 + A^3 B_3 = A(1)B(1) + A(2)B(2) + A(3)B(3)$, a scalar quantity. The physical component of the cross-product is $C(i) = h_i C^i = h_i \mathcal{E}^{ijk} A_j B_k$, no-summation on index i. Expanding the expression gives $C(1) = h_1 C^1 = h_1 \mathcal{E}^{1jk} A_j B_k = \dfrac{e_{1jk}}{\mathcal{J}} A_j B_k = \dfrac{1}{r}\left[h_2 A(2) h_3 B(3) - h_3 A(3) h_2 B(2)\right] = A(2)B(3) - A(3)B(2)$, and $C(2) = h_2 C^2 = r\mathcal{E}^{2jk} A_j B_k = r\dfrac{e_{2jk}}{\mathcal{J}} A_j B_k = \left[h_3 A(3) h_1 B(1) - h_1 A(1) h_3 B(3)\right] = A(3)B(1) - A(1)B(3)$, and $C(3) = h_3 C^3 = h_3 \mathcal{E}^{3jk} A_j B_k = \dfrac{e_{3jk}}{\mathcal{J}} A_j B_k = \dfrac{1}{r}\left[h_1 A(1) h_2 B(2) - h_2 A(2) h_1 B(1)\right] = A(1)B(2) - A(2)B(1)$.

b. For spherical coordinates, we have the scale factors as $(h_1, h_2, h_3) = (1, r, r\sin\varphi)$ and the Jacobian is $\mathcal{J} = r^2 \sin\varphi$, see section 11.6.1. We can write $B_1 = h_1 B(1) = B(1)$, $B_2 = h_2 B(2) = rA(2)$, $B_3 = h_3 B(3) = B(3)$. Similarly, $A^1 = \dfrac{A(1)}{h_1} = A(1)$, $A^2 = \dfrac{A(2)}{h_2} = \dfrac{A(2)}{r}$, $A^3 = \dfrac{A(3)}{h_3} = A(3)$. Therefore, the dot-product reads

$\vec{A} \cdot \vec{B} = A^i B_i = A^1 B_1 + A^2 B_2 + A^3 B_3 = A(1)B(1) + A(2)B(2) + A(3)B(3)$, a scalar quantity.

The physical component of the cross-product is $C(i) = h_i C^i = h_i \mathcal{E}^{ijk} A_j B_k$, no-summation on index i. Expanding the expression gives $C(1) = h_1 C^1 = h_1 \mathcal{E}^{1jk} A_j B_k = \dfrac{e_{1jk}}{\mathcal{J}} A_j B_k = \dfrac{1}{r^2 \sin\varphi}\left[h_2 A(2) h_3 B(3) - h_3 A(3) h_2 B(2)\right] =$

$A(2)B(3)-A(3)B(2)$, and $C(2)=h_2C^2=r\mathcal{E}^{2jk}A_jB_k=r\dfrac{e_{2jk}}{\mathcal{J}}A_jB_k=$ $\left[h_3A(3)h_1B(1)-h_1A(1)h_3B(3)\right]=A(3)B(1)-A(1)B(3)$, and $C(3)=$ $h_3C^3=h_3\mathcal{E}^{3jk}A_jB_k=r\sin\varphi\dfrac{e_{3jk}}{\mathcal{J}}A_jB_k=\dfrac{1}{r}\left[h_1A(1)h_2B(2)-h_2A(2)h_1B(1)\right]=$ $A(1)B(2)-A(2)B(1)$.

16.8 EXAMPLE: RELATION BETWEEN JACOBIAN AND METRIC TENSOR DETERMINANTS

Show that determinant g of metric tensor g_{ij} is equal to the square of Jacobian $\mathcal{J}$, in a coordinate system, when measured against Cartesian system.

Solution:

Let's consider two arbitrary system x^i and x'^i. We can write the metric tensor transformation as $g'_{ij}=\dfrac{\partial x^k}{\partial x'^i}\dfrac{\partial x^l}{\partial x'^j}g_{kl}$, since it is a doubly covariant tensor. Taking determinant of the both sides gives $\left|g'_{ij}\right|=\left|\dfrac{\partial x^k}{\partial x'^i}\right|\left|\dfrac{\partial x^l}{\partial x'^j}\right|\left|g_{kl}\right|$. But $\left|\dfrac{\partial x^k}{\partial x'^i}\right|$ or $\left|\dfrac{\partial x^l}{\partial x'^j}\right|$ is the Jacobian of transformation, hence $\left|g'_{ij}\right|=\mathcal{J}^2\left|g_{kl}\right|$. Rewriting the expression and using $\left|g_{kl}\right|=g$ and $\left|g'_{ij}\right|=g'$, we get $g'=\mathcal{J}^2g$, or $\mathcal{J}=\sqrt{\dfrac{g'}{g}}$. Now if the original x^i system is the Cartesian system, then $g_{cartesian}=1$ and we receive $g'=\mathcal{J}^2$.

16.9 EXAMPLE: DETERMINANT OF METRIC TENSORS USING DISPLACEMENT VECTORS

For an arbitrary coordinate system $x^i \equiv (\xi, \eta, \zeta)$ the magnitude of displacement vector is given by $ds^2 = 5d\xi^2 + 3d\eta^2 + 7d\zeta^2 - 4d\xi d\eta - 8d\xi d\zeta + 2d\eta d\zeta$. Find g and g^{ij} (i.e., the determinant of covariant metric tensor and the contravariant metric tensor).

Solution:

We have $ds^2 = g_{ij}dx^i dx^j$, expanding for the given coordinate system and using $g_{ij} = g_{ji}$, gives $ds^2 = g_{11}d\xi^2 + g_{22}d\eta^2 + g_{33}d\zeta^2 + 2g_{12}d\xi d\eta + 2g_{13}d\xi d\zeta + 2g_{23}d\eta d\zeta$. Therefore, after comparison we receive $g_{11} = 5$, $g_{22} = 3$, $g_{33} = 7$, $g_{12} = -2$, $g_{13} = -4$, and $g_{23} = 1$. The matrix form of covariant metric tensor is $g_{ij} = \begin{bmatrix} 5 & -2 & -4 \\ -2 & 3 & 1 \\ -4 & 1 & 7 \end{bmatrix}$. Determinant of $g_{ij} = g = 5(21-1) + 2(-14+4) - 4(-2+12) = 40$. Note that the given system is not orthogonal, since off-diagonal elements of g_{ij} are not zero. For contravariant metric tensor, we find the inverse of $g_{ij} = \begin{bmatrix} 5 & -2 & -4 \\ -2 & 3 & 1 \\ -4 & 1 & 7 \end{bmatrix}$, or $g^{ij} = \begin{bmatrix} 5 & -2 & -4 \\ -2 & 3 & 1 \\ -4 & 1 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{19}{40} & \frac{3}{40} \\ \frac{1}{4} & \frac{3}{40} & \frac{11}{40} \end{bmatrix}$.

This is calculated using the formula $g^{ij} = \frac{C_{ij}}{g} = \frac{cofactor\ of\ g_{ij}}{determinante\ of\ g_{ij}}$, for example; $g^{11} = \frac{C_{11}}{40} = \frac{\begin{vmatrix} 3 & 1 \\ 1 & 7 \end{vmatrix}}{40} = \frac{21-1}{40} = 20$, $g^{23} = \frac{C_{23}}{40} = \frac{-\begin{vmatrix} 5 & -2 \\ -4 & 1 \end{vmatrix}}{40} = \frac{-5+8}{40} = \frac{3}{40}$, etc.

16.10 EXAMPLE: DETERMINANT OF A 4 × 4 MATRIX USING PERMUTATION SYMBOLS

Calculate determinant of matrix $M = \begin{bmatrix} 5 & 8 & 15 & 21 \\ 4 & 10 & 13 & 11 \\ 3 & 6 & 9 & 12 \\ 2 & 4 & 7 & 1 \end{bmatrix}$, using Equation 9.13.

Solution:

We expand based on row, hence $|M| = e_{ijkl} M_{1i} M_{2j} M_{3k} M_{4l}$. Expanding we get $|M| =$

$$\begin{aligned}
&e_{1234}M_{11}M_{22}M_{33}M_{44} + e_{1243}M_{11}M_{22}M_{34}M_{43} + e_{1342}M_{11}M_{23}M_{34}M_{42} +\\
&e_{1324}M_{11}M_{23}M_{32}M_{44} + e_{1423}M_{11}M_{24}M_{32}M_{43} + e_{1432}M_{11}M_{24}M_{33}M_{42} +\\
&e_{2134}M_{12}M_{21}M_{33}M_{44} + e_{2143}M_{12}M_{21}M_{34}M_{43} + e_{2341}M_{12}M_{23}M_{34}M_{41} +\\
&e_{2314}M_{12}M_{23}M_{31}M_{44} + e_{2413}M_{12}M_{24}M_{31}M_{43} + e_{2431}M_{12}M_{24}M_{33}M_{41} +\\
&e_{3124}M_{13}M_{21}M_{32}M_{44} + e_{3142}M_{13}M_{21}M_{34}M_{42} + e_{3241}M_{13}M_{22}M_{34}M_{41} +\\
&e_{3214}M_{13}M_{22}M_{31}M_{44} + e_{3412}M_{13}M_{24}M_{31}M_{42} + e_{3421}M_{13}M_{24}M_{32}M_{41} +\\
&e_{4123}M_{14}M_{21}M_{32}M_{43} + e_{4132}M_{14}M_{21}M_{33}M_{42} + e_{4231}M_{14}M_{22}M_{33}M_{41} +\\
&e_{4213}M_{14}M_{22}M_{31}M_{43} + e_{4312}M_{14}M_{23}M_{31}M_{42} + e_{4321}M_{14}M_{23}M_{32}M_{41}.
\end{aligned}$$

Substituting for the values of the matrix elements and the permutation symbols (i.e., $+1$ for even and -1 for odd number of interchanges for indices), we receive $|M| = 18$. Note that the number of terms out of expansion is equal to $4! = 4 \times 3 \times 2 = 24$, or in general $N!$ for an $N \times N$ matrix.

16.11 EXAMPLE: TIME DERIVATIVES OF THE JACOBIAN

Show that the time derivative of the Jacobian reads $\frac{1}{\mathcal{J}}\frac{\partial \mathcal{J}}{\partial t} = \vec{\nabla} \cdot \vec{V}$, where $\vec{V}$ is the velocity vector.

Solution:

Considering the transformation from coordinate system x^i to that of x'^i, we can write the Jacobian as the determinant of the transformation, using Equation 4.4, or $\mathcal{J} = \varepsilon^{knl} \frac{\partial x^1}{\partial x'^k}\frac{\partial x^2}{\partial x'^n}\frac{\partial x^3}{\partial x'^l}$. Taking the time derivative, we have

$$\frac{\partial \mathcal{J}}{\partial t} = \varepsilon^{knl}\left[\frac{\partial}{\partial x'^k}\left(\frac{\partial x^1}{\partial t}\right)\frac{\partial x^2}{\partial x'^n}\frac{\partial x^3}{\partial x'^l} + \frac{\partial x^1}{\partial x'^k}\frac{\partial}{\partial x'^n}\left(\frac{\partial x^2}{\partial t}\right)\frac{\partial x^3}{\partial x'^l} + \frac{\partial x^1}{\partial x'^k}\frac{\partial x^2}{\partial x'^n}\frac{\partial}{\partial x'^l}\left(\frac{\partial x^3}{\partial t}\right)\right].$$

But we can write $\dfrac{\partial x^1}{\partial t}=V^1, \dfrac{\partial x^2}{\partial t}=V^2, \dfrac{\partial x^3}{\partial t}=V^3$, i.e., contravariant components of velocity vector. Hence, using chain-rule or $\dfrac{\partial x^i}{\partial x'^n}=\dfrac{\partial x^i}{\partial x^j}\dfrac{\partial x^j}{\partial x'^n}$ we can write $\dfrac{\partial \mathcal{J}}{\partial t}=$

$$\varepsilon^{knl}\left[\underbrace{\frac{\partial V^1}{\partial x^i}\frac{\partial x^i}{\partial x'^k}\frac{\partial x^2}{\partial x^j}\frac{\partial x^j}{\partial x'^n}\frac{\partial x^3}{\partial x^p}\frac{\partial x^p}{\partial x'^l}}_{\neq 0;\, i=1\ \&\, j=2\ \&\, p=3}+\underbrace{\frac{\partial x^1}{\partial x^j}\frac{\partial x^j}{\partial x'^k}\frac{\partial V^2}{\partial x^i}\frac{\partial x^i}{\partial x'^n}\frac{\partial x^3}{\partial x^p}\frac{\partial x^p}{\partial x'^l}}_{\neq 0;\, j=1\ \&\, i=2\ \&\, p=3}+\underbrace{\frac{\partial x^1}{\partial x^j}\frac{\partial x^j}{\partial x'^k}\frac{\partial x^2}{\partial x^i}\frac{\partial x^i}{\partial x'^n}\frac{\partial V^3}{\partial x^p}\frac{\partial x^p}{\partial x'^l}}_{\neq 0;\, j=1\ \&\, i=2\ \&\, p=3}\right].$$

Therefore, after rearranging the terms, in each group, we receive

$$\frac{\partial \mathcal{J}}{\partial t}=\frac{\partial V^1}{\partial x^1}\left(\varepsilon^{knl}\frac{\partial x^1}{\partial x'^k}\frac{\partial x^2}{\partial x'^n}\frac{\partial x^3}{\partial x'^l}\right)+\frac{\partial V^2}{\partial x^2}\left(\varepsilon^{knl}\frac{\partial x^1}{\partial x'^k}\frac{\partial x^2}{\partial x'^n}\frac{\partial x^3}{\partial x'^l}\right)+\frac{\partial V^3}{\partial x^3}\left(\varepsilon^{knl}\frac{\partial x^1}{\partial x'^k}\frac{\partial x^2}{\partial x'^n}\frac{\partial x^3}{\partial x'^l}\right).$$

But the terms in each bracket is the Jacobian, and hence we receive $\dfrac{\partial \mathcal{J}}{\partial t}=\mathcal{J}\left(\dfrac{\partial V^1}{\partial x^1}+\dfrac{\partial V^2}{\partial x^2}+\dfrac{\partial V^3}{\partial x^3}\right)=\mathcal{J}\underbrace{\left[\dfrac{1}{\mathcal{J}}\left(\mathcal{J}\dfrac{\partial V^i}{\partial x^i}\right)\right]}_{\vec{\nabla}\cdot\vec{V}}=\mathcal{J}\vec{\nabla}\cdot\vec{V}$, which yields $\dfrac{1}{\mathcal{J}}\dfrac{\partial \mathcal{J}}{\partial t}=\vec{\nabla}\cdot\vec{V}$. For divergence of a vector, see Equation 11.12.

16.12 EXAMPLE: COVARIANT DERIVATIVES OF A CONSTANT VECTOR

Show that for a constant vector $\vec{C}$, the covariant derivative $C^i_{,j}$ is not necessarily zero. Find the expression for its derivative.

Solution:

We have $\vec{C}=C^i\vec{e}_i$; writing its covariant derivative, we receive $\dfrac{\partial}{\partial x^j}\left(C^i\vec{e}_i\right)\vec{e}^j$.

The covariant component then reads as $\dfrac{\partial}{\partial x^j}\left(C^i\vec{e}_i\right)=\left(\dfrac{\partial C^i}{\partial x^j}\right)\vec{e}_i+C^i\dfrac{\partial}{\partial x^j}\left(\vec{e}_i\right)=$ $\left(\dfrac{\partial C^i}{\partial x^j}\right)\vec{e}_i+\underbrace{C^i\Gamma^k_{ij}\vec{e}_k}_{i\leftrightarrow k}=\vec{e}_i\underbrace{\left(\dfrac{\partial C^i}{\partial x^j}+C^k\Gamma^i_{kj}\right)}_{C,^i_j}$. Now, since $\vec{C}$ is a constant, the first term in the bracket vanishes but the second term does not. Therefore, we receive $C^i_{,j}=C^k\Gamma^i_{kj}$.

16.13 EXAMPLE: COVARIANT DERIVATIVES OF PHYSICAL COMPONENTS OF A VECTOR

For the vector $\vec{A} = A^i\vec{e}_i = A(i)\vec{e}(i)$, write down the expression for $A(i)_{,j}$. Use the covariant derivative of the vector $\vec{A}$ in terms its physical components, or $\nabla_j\vec{A} = \nabla_j\left(A(i)\vec{e}(i)\right) = \vec{e}(i)A(i)_{,j}$.

Solution:

The covariant derivative, in terms of contravariant component of the vector is $\nabla_j\vec{A} = \nabla_j\left(A^i\vec{e}_i\right) = \left(\dfrac{\partial A^i}{\partial x^j} + A^k\Gamma^i_{kj}\right)\vec{e}_i$. Substituting for contravariant components $A^i = A(i)/h_i$ and covariant basis vector $\vec{e}_i = h_i\vec{e}(i)$, we get $\nabla_j\vec{A} =$

$$\left(\frac{\partial\left(\frac{A(i)}{h_i}\right)}{\partial x^j} + \frac{A(k)}{h_k}\Gamma^i_{kj}\right)h_i\vec{e}(i) = \left(\frac{\partial\left(A(i)\right)}{\partial x^j} - \frac{A(i)}{h_i}\frac{\partial\left(h_i\right)}{\partial x^j} + \frac{h_i}{h_k}A(k)\Gamma^i_{kj}\right)\vec{e}(i).$$

Therefore, $A(i)_{,j} = \dfrac{\partial\left(A(i)\right)}{\partial x^j} - \dfrac{A(i)}{h_i}\dfrac{\partial\left(h_i\right)}{\partial x^j} + \dfrac{h_i}{h_k}A(k)\Gamma^i_{kj} = \nabla_j A(i) - \dfrac{A(i)}{h_i}\nabla_j h_i + \dfrac{h_i}{h_k}A(k)\Gamma^i_{kj}$. Note that the Christoffel symbol could be written in terms of physical components/ Unit vectors, i.e., Γ^i_{kj} could be written in terms of $\vec{e}(i)$, designated by $\hat{\Gamma}^i_{kj}$ (see Example 10.4.1).

16.14 EXAMPLE: CONTINUITY EQUATIONS IN SEVERAL COORDINATE SYSTEMS

The continuity for a fluid with density ρ and velocity V^i is given as $\dfrac{\partial\rho}{\partial t} + \dfrac{1}{\mathcal{J}}\dfrac{\partial}{\partial x^i}\left(\mathcal{J}\rho V^i\right) = 0$. Write this equation in tensor notation for $N = 3$, and express the continuity equation in:

a. Cartesian coordinates (x, y, z) with velocity physical components given as $\vec{V} = (V_x, V_y, V_z)$.

b. Cylindrical coordinates (r,θ,z) with velocity physical components given as $\vec{V}=(V_r,V_\theta,V_z)$.

c. Spherical coordinates (r,φ,θ) with velocity physical components given as $\vec{V}=(V_r,V_\varphi,V_\theta)$.

Solution:

Summation on i gives, $\dfrac{\partial\rho}{\partial t}+\dfrac{1}{\mathcal{J}}\left[\dfrac{\partial}{\partial x^1}\left(\mathcal{J}\rho V^1\right)+\dfrac{\partial}{\partial x^2}\left(\mathcal{J}\rho V^2\right)+\dfrac{\partial}{\partial x^3}\left(\mathcal{J}\rho V^3\right)\right]=0.$

a. Jacobian for a Cartesian system is unity and physical and contravariant components are identical, hence the continuity equation reads,

$$\frac{\partial\rho}{\partial t}+\left[\frac{\partial}{\partial x}(\rho V_x)+\frac{\partial}{\partial y}\left(\rho V_y\right)+\frac{\partial}{\partial z}(\rho V_z)\right]=0.$$

b. Jacobian for cylindrical systems is $\mathcal{J}=r$ and $V^1=V_r$, $V^2=V_\theta/r$, $V^3=V_z$. Hence, the continuity reads, after some manipulations,

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\rho V_r)+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho V_\theta)+\frac{\partial}{\partial z}(\rho V_z)=0.$$

c. Jacobian for spherical systems is $\mathcal{J}=r^2\sin\varphi$ and $V^1=V_r$, $V^2=V_\varphi/r$, $V^3=V_\theta/\left(r\sin\varphi\right)$. Hence, the continuity reads, after some manipulations,

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2\sin\varphi}\left[\frac{\partial}{\partial r}\left(r^2\sin\varphi\rho V_r\right)+\frac{\partial}{\partial\varphi}\left(r\sin\varphi\rho V_\varphi\right)+\frac{\partial}{\partial\theta}\left(r\rho V_\theta\right)\right]=0.$$

16.15 EXAMPLE: 4D SPHERICAL COORDINATE SYSTEMS

A 4D spherical system $x^i\equiv(r,\psi,\theta,\varphi)$ with two polar angles ψ and θ range from 0 to π and azimuth angle φ from 0 to 2π, is related to the 4D Cartesian coordinates $y^i\equiv(x,y,z,w)$ as

$$\begin{cases} x=r\sin\psi\sin\theta\cos\varphi \\ y=r\sin\psi\sin\theta\sin\varphi \\ z=r\sin\psi\cos\theta \\ w=r\cos\psi \end{cases}$$

a. Find the Jacobian for the spherical system and volume of a 4D sphere in this coordinate system.

b. Find the expression for the Laplacian of a scalar in the spherical coordinates and write out it in detail.

Solution:

1. Jacobian reads $\mathcal{J}=\dfrac{\partial(x,y,z,w)}{\partial(r,\psi,\theta,\varphi)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \psi} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \psi} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \psi} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi}\\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial \psi} & \frac{\partial w}{\partial \theta} & \frac{\partial w}{\partial \varphi}\end{vmatrix}=$

$$\begin{vmatrix}\sin\psi\sin\theta\cos\varphi & r\cos\psi\sin\theta\cos\varphi & r\sin\psi\sin\theta\cos\varphi & -r\sin\psi\sin\theta\sin\varphi\\ \sin\psi\sin\theta\sin\varphi & r\cos\psi\sin\theta\sin\varphi & r\sin\psi\cos\theta\sin\varphi & r\sin\psi\sin\theta\cos\varphi\\ \sin\psi\cos\theta & r\cos\psi\cos\theta & -r\sin\psi\sin\theta & 0\\ \cos\psi & -r\sin\psi & 0 & 0\end{vmatrix}=$$

$-r^3\sin^2\psi\sin\theta.$

The differential volume is $d\mathbb{V}=\mathcal{J}dr d\psi d\theta d\varphi$, with $0\le\theta$ & $\psi\le\pi$ and $0\le\varphi\le 2\pi$. Volume is then a quadruplet integral of $d\mathbb{V}$, or

$$\mathbb{V}=\int_0^R(-r^3)\int_0^\pi\sin^2\psi d\psi\int_0^\pi\sin\theta d\theta\int_0^{2\pi}d\varphi=\pi R^4\int_0^\pi\frac{1-\cos 2\varphi}{2}=\frac{\pi^2R^4}{2}.$$

2. The expression for the Laplacian of scalar S reads $\nabla^2 S=\dfrac{1}{\mathcal{J}}\nabla_i\left(\mathcal{J}g^{ij}\nabla_j S\right)$. This coordinate system is orthogonal, since $g_{ij}=0$ for $i\neq j$. For example $g_{12}=\vec{e}_1\cdot\vec{e}_2=r\sin\psi\cos\psi\sin^2\theta\cos^2\theta-r\sin\psi\cos\psi\sin^2\theta\sin^2\varphi+r\sin\psi\cos\psi$ $\cos^2\theta-r\sin\psi\cos\psi=0$. Therefore, $g_{ii}=\dfrac{1}{g^{ii}}=h_i^2$. For calculating $g_{ii}=\vec{e}_i\cdot\vec{e}_i$, need the covariant basis vectors, or

$$\begin{cases}\vec{e}_1=\sin\psi\sin\theta\cos\varphi\vec{E}_1+\sin\psi\sin\theta\sin\varphi\vec{E}_2+\sin\psi\cos\theta\vec{E}_3+\cos\psi\vec{E}_4\\ \vec{e}_2=r\cos\psi\sin\theta\cos\varphi\vec{E}_1+r\cos\psi\sin\theta\sin\varphi\vec{E}_2+r\cos\psi\cos\theta\vec{E}_3-r\sin\psi\vec{E}_4\\ \vec{e}_3=r\sin\psi\cos\theta\cos\varphi\vec{E}_1+r\sin\psi\cos\theta\sin\varphi\vec{E}_2-r\sin\psi\sin\theta\vec{E}_3\\ \vec{e}_4=-r\sin\psi\sin\theta\sin\varphi\vec{E}_1+r\sin\psi\sin\theta\cos\varphi\vec{E}_2\end{cases}$$

which yields, $g_{11}=1$, $g_{22}=r^2$, $g_{33}=r^2\sin^2\psi$, $g_{44}=r^2\sin^2\psi\sin^2\theta$. Now, $\nabla^2 S=\frac{1}{\mathcal{J}}\nabla_i\left(\mathcal{J}g^{ij}\nabla_j S\right)=\nabla^2 S=\frac{1}{\mathcal{J}}\nabla_i\left(\mathcal{J}g^{ii}\nabla_i S\right)=\frac{1}{\mathcal{J}}\nabla_1\left(\mathcal{J}g^{11}\nabla_1 S\right)+\frac{1}{\mathcal{J}}\nabla_2\left(\mathcal{J}g^{22}\nabla_2 S\right)+\frac{1}{\mathcal{J}}\nabla_3\left(\mathcal{J}g^{33}\nabla_3 S\right)+\frac{1}{\mathcal{J}}\nabla_4\left(\mathcal{J}g^{44}\nabla_4 S\right)$. Each term reads, after substitution

$$\frac{1}{\mathcal{J}}\nabla_1\left(\mathcal{J}g^{11}\nabla_1 S\right)=\frac{-1}{r^3\sin^2\psi\sin\theta}\frac{\partial}{\partial r}\left(-r^3\sin^2\psi\sin\theta\frac{\partial S}{\partial r}\right)=\frac{3}{r}\frac{\partial S}{\partial r}+\frac{\partial^2 S}{\partial r^2}$$

$$\begin{aligned}\frac{1}{\mathcal{J}}\nabla_2\left(\mathcal{J}g^{22}\nabla_2 S\right)&=\frac{-1}{r^3\sin^2\psi\sin\theta}\frac{\partial}{\partial\psi}\left(-r^{-2}r^3\sin^2\psi\sin\theta\frac{\partial S}{\partial\psi}\right)\\&=\frac{1}{r^2}\left(\cot\psi\frac{\partial S}{\partial\psi}+\frac{\partial^2 S}{\partial\psi^2}\right)\end{aligned}$$

$$\begin{aligned}\frac{1}{\mathcal{J}}\nabla_3\left(\mathcal{J}g^{33}\nabla_3 S\right)&=\frac{-1}{r^3\sin^2\psi\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{-r^3\sin^2\psi\sin\theta}{r^2\sin^2\psi}\frac{\partial S}{\partial\theta}\right)\\&=\frac{1}{r^2\sin^2\psi}\left(\cot\theta\frac{\partial S}{\partial\theta}+\frac{\partial^2 S}{\partial\theta^2}\right)\end{aligned}$$

$$\begin{aligned}\frac{1}{\mathcal{J}}\nabla_4\left(\mathcal{J}g^{44}\nabla_4 S\right)&=\frac{-1}{r^3\sin^2\psi\sin\theta}\frac{\partial}{\partial\varphi}\left(\frac{-r^3\sin^2\psi\sin\theta}{r^2\sin^2\psi\sin^2\theta}\frac{\partial S}{\partial\varphi}\right)\\&=\frac{1}{r^2\sin^2\psi\sin^2\theta}\frac{\partial^2 S}{\partial\varphi^2}\end{aligned}$$

Hence, $\nabla^2 S=\frac{3}{r}\frac{\partial S}{\partial r}+\frac{\partial^2 S}{\partial r^2}+\frac{1}{r^2}\left(\cot\psi\frac{\partial S}{\partial\psi}+\frac{\partial^2 S}{\partial\psi^2}\right)+\frac{1}{r^2\sin^2\psi}\left(\cot\theta\frac{\partial S}{\partial\theta}+\frac{\partial^2 S}{\partial\theta^2}\right)+\frac{1}{r^2\sin^2\psi\sin^2\theta}\frac{\partial^2 S}{\partial\varphi^2}$.

16.16 EXAMPLE: COMPLEX DOUBLE DOT-CROSS PRODUCT EXPRESSIONS

The expression $\underbrace{\vec{\nabla}}_{\times}\overbrace{\vec{A}}^{\times}\,\underbrace{\vec{\nabla}}_{\times}\overbrace{\vec{A}}^{\times}$ is given. The notations mean that the two gradient vectors (marked by $\times$) are cross product to each other as well as the two vectors $\vec{A}$, then the resulted vectors of the cross-product operations are dotted. Write the expression in tensor notation and simplify the result.

Solution:

The two cross-product results are dotted; hence we require the contravariant component of the cross-product of the gradient's and the covariant component of the cross-product of the vectors $\vec{A}$. Therefore, we can write $\underbrace{\vec{\nabla}}_{\times}\overbrace{\vec{A}}^{\times}\,\underbrace{\vec{\nabla}}_{\times}\overbrace{\vec{A}}^{\times} = \varepsilon^{ijk}\left(\varepsilon_{ilm}A^{l}A_{,k}^{m}\right)_{,j}$. Expanding the resulted expression yields, $\varepsilon^{ijk}\varepsilon_{ilm}\left(A^{l}A_{,k}^{m}\right)_{,j} = \delta_{l}^{j}\delta_{m}^{k}\left(A^{l}A_{,k}^{m}\right)_{,j} - \delta_{m}^{j}\delta_{l}^{k}\left(A^{l}A_{,k}^{m}\right)_{,j} = \left(A^{l}A_{,k}^{k}\right)_{,l} - \left(A^{k}A_{,k}^{j}\right)_{,j}$.

Performing the covariant derivative operations gives $\underbrace{\vec{\nabla}}_{\times}\overbrace{\vec{A}}^{\times}\,\underbrace{\vec{\nabla}}_{\times}\overbrace{\vec{A}}^{\times} = A_{,l}^{l}A_{,k}^{k} + A^{l}A_{,kl}^{k} - A_{,j}^{k}A_{,k}^{j} - \underbrace{A^{k}A_{,kj}^{j}}_{k\to l,\, j\to k} = A_{,l}^{l}A_{,k}^{k} - A_{,j}^{k}A_{,k}^{j}$. The first term of the result can be written in vector notation as $A_{,l}^{l}A_{,k}^{k} = \left(\vec{\nabla}\cdot\vec{A}\right)^{2}$. Therefore, we finally have $\underbrace{\vec{\nabla}}_{\times}\overbrace{\vec{A}}^{\times}\,\underbrace{\vec{\nabla}}_{\times}\overbrace{\vec{A}}^{\times} = \left(\vec{\nabla}\cdot\vec{A}\right)^{2} - A_{,j}^{k}A_{,k}^{j}$. Using the method given in this example, apply it to the following expressions:

a. $\overbrace{\vec{\nabla}}^{\times}\underbrace{\vec{A}}_{\times}\underbrace{\vec{\nabla}}_{\times}\overbrace{\vec{A}}^{\times} = \varepsilon^{ijk}\left(\varepsilon_{ilm}A^{l}A_{k,\cdot}{}^{m}\right)_{,j} = \delta_{l}^{j}\delta_{m}^{k}\left(A^{l}A_{k,\cdot}{}^{m}\right)_{,j} - \delta_{m}^{j}\delta_{l}^{k}\left(A^{l}A_{k,\cdot}{}^{m}\right)_{,j} = \left(A^{j}A_{k,\cdot}{}^{k}\right)_{,j} - \left(A^{k}A_{k,\cdot}{}^{j}\right)_{,j} = A_{,j}^{j}A_{k,\cdot}{}^{k} + A^{j}A_{k,\cdot j}{}^{k} - A_{,j}^{k}A_{k,\cdot}{}^{j} - A^{k}A_{k,\cdot j}{}^{j}$. In vector notation, the resulted expression can be written as $\overbrace{\vec{\nabla}}^{\times}\underbrace{\vec{A}}_{\times}\,\underbrace{\vec{\nabla}}_{\times}\overbrace{\vec{A}}^{\times} = \left(\vec{\nabla}\cdot\vec{A}\right)^{2} + \vec{A}\cdot\left(\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)\right) - A_{,j}^{k}A_{k,\cdot}{}^{j} - \vec{A}\cdot\left(\nabla^{2}\vec{A}\right)$.

b. $\underbrace{\vec{\nabla}}_{\times}\overbrace{\vec{\nabla}}^{\times}\underbrace{\vec{A}}_{\times}\overbrace{\vec{A}}^{\times}=\varepsilon^{ijk}\varepsilon_{ilm}\left(A^m A_k\right)_{,\cdot j}^{\;\;\,l}=\left(\delta_l^j\delta_m^k-\delta_m^j\delta_l^k\right)\left(A^m A_k\right)_{,\cdot j}^{\;\;\,l}=\left(A^k A_k\right)_{,\cdot j}^{\;\;\,j}-$

$$\left(A^j A_k\right)_{,\cdot j}^{\;\;\,k}=\nabla^2 A^2-2\vec{A}\cdot\left(\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)\right)-A_{,\cdot}^{j\,k}A_{k,j}-\left(\vec{\nabla}\cdot\vec{A}\right)^2.$$

Note that vector-type notation is not sufficient to express some of the terms in these expressions—another reason to use index/tensor notation for its generality.

16.17 EXAMPLE: COVARIANT DERIVATIVES OF METRIC TENSORS

Show that the derivative of metric tensors (i.e., g_{ij} and g^{ij}) is equal to zero.

Solution:

Since metric tensor is a constant in the Cartesian coordinate system, hence its derivative is null and will be the same for any coordinate system. To show this, we use Equation 10.14, or $\left(g_{ij}\right)_{,k}=\dfrac{\partial g_{ij}}{\partial x^k}-\Gamma_{ik}^{n}\,g_{nj}-\Gamma_{jk}^{n}g_{ni}$. Now using Equation 10.19 we can expand the last two terms on the R.H.S, or $\left(g_{ij}\right)_{,k}=\dfrac{\partial g_{ij}}{\partial x^k}-\dfrac{1}{2}\left(\dfrac{\partial g_{ij}}{\partial x^k}+\dfrac{\partial g_{kj}}{\partial x^i}-\dfrac{\partial g_{ki}}{\partial x^j}\right)-\dfrac{1}{2}\left(\dfrac{\partial g_{ij}}{\partial x^k}+\dfrac{\partial g_{ki}}{\partial x^j}-\dfrac{\partial g_{kj}}{\partial x^i}\right)$. The terms on the R.H.S cancel out and the expression is equal to zero, hence $\left(g_{ij}\right)_{,k}=0$, and similarly $\left(g^{ij}\right)_{,k}=0$.

16.18 EXAMPLE: ACTIVE ROTATION USING SINGLE-AXIS AND QUATERNIONS METHODS

Consider the vector $\overrightarrow{OA}=(1,-1,2)$ connecting the origin to point A in the (x,y,z) coordinate system. Now we rotate this vector by an angle of 60° about an axis in the direction of unit vector $\vec{n}=\left(0,\dfrac{1}{2},\dfrac{\sqrt{3}}{2}\right)$. Calculate the resulted vector $\overrightarrow{OA'}$, using a) rotation matrix with Equation 12.7, and b) quaternion method with Equation 15.12. Note that the rotation angle here is given for an active rotation (i.e., vector rotates but coordinate system is kept fix).

Solution:

a. Since the vector $\overrightarrow{OA}$ is rotated by 60 degrees, active rotation, then we use $\theta = -60^\circ$ in Equation 12.7. Therefore, for $n_1 = 0$, $n_2 = 1/2$, and $n_3 = \sqrt{3}/2$ we receive $R = \begin{bmatrix} \frac{1}{2} & -\frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{3}{4} & \frac{5}{8} & \frac{\sqrt{3}}{8} \\ -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{8} & \frac{7}{8} \end{bmatrix}$.

The rotated vector component in the (x, y, z) is given by

$$\begin{bmatrix} \frac{1}{2} & -\frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{3}{4} & \frac{5}{8} & \frac{\sqrt{3}}{8} \\ -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{8} & \frac{7}{8} \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \\ 2 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2} + \frac{3}{4} + \frac{\sqrt{3}}{2} \\ \frac{3}{4} - \frac{5}{8} + \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{8} + \frac{7}{4} \end{Bmatrix} = \frac{1}{8} \begin{Bmatrix} 10 + 4\sqrt{3} \\ 1 + 2\sqrt{3} \\ 14 - 3\sqrt{3} \end{Bmatrix}, \qquad \text{or}$$

$$\overrightarrow{OA'} = \left(\frac{10 + 4\sqrt{3}}{8}, \frac{1 + 2\sqrt{3}}{8}, \frac{14 - 3\sqrt{3}}{8} \right).$$

b. Using the active quaternion operator, see Equation 15.12, we need to have the corresponding quaternion based on half of the rotation angle,

$\frac{60^\circ}{2} = 30^\circ$, $Q = \cos 30 + \underbrace{\left(\frac{1}{2} j + \frac{\sqrt{3}}{2} k \right)}_{\vec{n}} \sin 30 = \frac{\sqrt{3}}{2} + \frac{1}{4} j + \frac{\sqrt{3}}{4} k$. Now, we calculate $\overrightarrow{OA'} = Q\overrightarrow{OA}Q^* = \left(\frac{\sqrt{3}}{2} + \frac{1}{4} j + \frac{\sqrt{3}}{4} k \right)(i - j + 2k)\left(\frac{\sqrt{3}}{2} - \frac{1}{4} j - \frac{\sqrt{3}}{4} k \right) =$

$$\left[\frac{1}{4}-\frac{\sqrt{3}}{2}+\left(\frac{1}{2}+\frac{3\sqrt{3}}{4}\right)i-\frac{\sqrt{3}}{4}j-\left(\frac{1}{4}-\sqrt{3}\right)k\right](i-j+2k)=\frac{10+4\sqrt{3}}{8}i+$$

$$\frac{1+2\sqrt{3}}{8}j+\frac{14-3\sqrt{3}}{8}k. \quad \text{Or} \quad \overline{OA'}=\left(\frac{10+4\sqrt{3}}{8},\frac{1+2\sqrt{3}}{8},\frac{14-3\sqrt{3}}{8}\right).$$

Alternatively, we can use Equation 15.14. Readers may want to use Wolfram Alpha (*www.wolframalpha.com*) to visualize the quaternion as a rotation operator. After accessing the wolfram Web site, enter the phrase "quaternions" in the space provided and use the format provided similar to those examples given on the page.

16.19 EXAMPLE: PASSIVE ROTATION USING SINGLE-AXIS AND QUATERNIONS METHODS

Repeat Example 16.18 for a passive rotation of an angle 45°. Having passive rotation we keep vector $\overrightarrow{OA}$ fixed and rotate coordinates (x,y,z). Repeating the data here, for convenience, $\overrightarrow{OA}=(1,-1,2)$, $\vec{n}=\left(0,\frac{1}{2},\frac{\sqrt{3}}{2}\right)$.

Solution:

a. Since the vector $\overrightarrow{OA}$ is rotated by 45 degrees, passive rotation, then we use $\theta=45^{\circ}$ in Equation 12.7. Therefore, for $n_1=0$, $n_2=1/2$, and $n_3=\sqrt{3}/2$ we receive $R=\begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{4} & -\frac{\sqrt{2}}{4}\\ -\frac{\sqrt{6}}{4} & \frac{2+3\sqrt{2}}{8} & \frac{2\sqrt{3}-\sqrt{6}}{8}\\ \frac{\sqrt{2}}{4} & \frac{2\sqrt{3}-\sqrt{6}}{8} & \frac{6+\sqrt{2}}{8}\end{bmatrix}$. The vector components in the rotated coordinates (x',y',z') is given by

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{4} & -\frac{\sqrt{2}}{4} \\ -\frac{\sqrt{6}}{4} & \frac{2+3\sqrt{2}}{8} & \frac{2\sqrt{3}-\sqrt{6}}{8} \\ \frac{\sqrt{2}}{4} & \frac{2\sqrt{3}-\sqrt{6}}{8} & \frac{6+\sqrt{2}}{8} \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \\ 2 \end{Bmatrix} = \begin{Bmatrix} \frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{6}}{4} - \frac{2+3\sqrt{2}}{8} + \frac{2\sqrt{3}-\sqrt{6}}{4} \\ \frac{\sqrt{2}}{4} - \frac{2\sqrt{3}-\sqrt{6}}{8} + \frac{6+\sqrt{2}}{4} \end{Bmatrix} =$$

$$\frac{1}{8}\begin{Bmatrix} -2\sqrt{6} \\ -2-3\sqrt{2}+4\sqrt{3}-4\sqrt{6} \\ 12+4\sqrt{2}-2\sqrt{3}+\sqrt{6} \end{Bmatrix}, \text{ or } \overrightarrow{OA'} = \left(\frac{-2\sqrt{6}}{8}, \frac{-2-3\sqrt{2}+4\sqrt{3}-4\sqrt{6}}{8}, \right.$$

$$\left. \frac{12+4\sqrt{2}-2\sqrt{3}+\sqrt{6}}{8} \right) \cong (-0.61237, -1.13905, 2.0803).$$

b. The quaternion related to this rotation, using half of the rotation angle, is $Q = \cos\frac{\pi}{8} + \underbrace{\left(\frac{1}{2}j + \frac{\sqrt{3}}{2}k\right)}_{\vec{n}} \sin\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{4}j + \frac{\sqrt{6-3\sqrt{2}}}{4}k.$

Now, we calculate $(\overrightarrow{OA})' = Q^* \overrightarrow{OA} Q = \left(\frac{\sqrt{2+\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{4} j - \frac{\sqrt{6-3\sqrt{2}}}{4} k \right) (i - j + 2k) \left(\frac{\sqrt{2+\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{4} j + \frac{\sqrt{6-3\sqrt{2}}}{4} k \right) \cong.$ $(0.92388 - 0.191342j - 0.331414k)(i - j + 2k)(0.92389 + 0.191342j + 0.331414k) = (0.47149 + 0.20979i - 1.2553j + 2.03912k)(0.92389 + 0.191342j + 0.331414k) = -0.61237i - 1.13905j + 2.0803k.$ Or $(\overrightarrow{OA})' = (-0.61237, -1.13907, 2.08032)$. Alternatively, we can use Equation 15.20.

16.20 EXAMPLE: SUCCESSIVE ROTATIONS USING QUATERNIONS METHOD

Use data given in Example 16.18 and rotate the resulted vector, actively in that example through a successive rotation of an angle 45°. For this operation use the quaternion method.

Solution:

From Example 16.18 we have the resulted vector $\overrightarrow{OA'} = \left(\frac{10+4\sqrt{3}}{8}, \frac{1+2\sqrt{3}}{8}, \frac{14-3\sqrt{3}}{8} \right)$ and the unit vector $\vec{n} = \left(0, \frac{1}{2}, \frac{\sqrt{3}}{2} \right)$ along the axis of rotation. The follow up rotation quaternion is related to rotation angle of $45^{\circ} = \frac{\pi}{4}$. Hence, using half-angle we have $P = \cos\frac{\pi}{8} + \underbrace{\left(\frac{1}{2} j + \frac{\sqrt{3}}{2} k \right)}_{\vec{n}} \sin\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{4} j + \frac{\sqrt{6-3\sqrt{2}}}{4} k$. Now, using Equation 15.18 we can write $\overrightarrow{OA''} = P\overrightarrow{OA'}P^{*} =$

$$\left(\frac{\sqrt{2+\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{4} j + \frac{\sqrt{6-3\sqrt{2}}}{4} k \right) \left(\frac{10+4\sqrt{3}}{8} i + \frac{1+2\sqrt{3}}{8} j + \frac{14-3\sqrt{3}}{8} k \right)$$
$$\left(\frac{\sqrt{2+\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{4} j - \frac{\sqrt{6-3\sqrt{2}}}{4} k \right) \cong 1.5436i + 1.8708j + 0.3425k.$$

We can also use the original vector $\overrightarrow{OA}$ with the dual-quaternion as the product of the two corresponding quaternions, see Equation 15.18, or $(PQ)\overrightarrow{OA}(PQ)^{*}$. Note that the q belongs to the first rotation and p (60° rotation) to the second one (45° rotation). But $\hat{p}\hat{q} = \left(\frac{\sqrt{2+\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{4} j + \frac{\sqrt{6-3\sqrt{2}}}{4} k \right) \left(\frac{\sqrt{3}}{2} + \frac{1}{4} j + \frac{\sqrt{3}}{4} k \right) \cong 0.60876 + 0.39668j + 0.68706k$. Therefore, its conjugate $(\hat{p}\hat{q})^{*} = 0.60876 - 0.39668j - 0.68706k$. Now, we can write $\overrightarrow{OA''} = (\hat{p}\hat{q})\overrightarrow{OA}(\hat{p}\hat{q})^{*} = (0.60876 + 0.39668j + 0.68706k)\quad (i - j + 2k)(0.60876 - 0.39668j - 0.68706k) = 1.5436i + 1.8708j + 0.3425k$. As seen the two methods provide us with identical results.

CHAPTER 17

EXERCISE PROBLEMS

1. Write out the following expression in full detail for $N = 3$ dimensions using Einstein summation convention:

   a. $A = A^{ij}\vec{e}_i\vec{e}_j$

   b. $\vec{A}\cdot\vec{B} = A^i B^j g_{ij}$

   c. $\nabla^2\psi = \dfrac{1}{\mathcal{J}}\dfrac{\partial}{\partial x^i}\left(\mathcal{J}g^{ij}\dfrac{\partial\psi}{\partial x^j}\right)$

   d. $C(i) = h_i\dfrac{e_{ijk}}{\mathcal{J}}\dfrac{\partial\left(h_k A(k)\right)}{\partial x^j}$

   e. $A^{ij}_{k,n} = \dfrac{\partial A^{ij}_k}{\partial x^n} + \Gamma^i_{nm}A^{jm}_k + \Gamma^j_{nm}A^{im}_k - \Gamma^m_{nk}A^{ij}_m$

2. By rotating bipolar coordinates about X-axis (the axis where foci are located on) we obtain the bi-spherical coordinate system, $\left(x^1,x^2,x^3\right)\equiv(\xi,\eta,\varphi)$ given in terms of Cartesian $\left(y^1,y^2,y^3\right)\equiv(X,Y,Z)$, as

$$\begin{cases} X = \dfrac{a\sin\xi\cos\varphi}{\cosh\eta - \cos\xi} \\ Y = \dfrac{a\sin\xi\sin\varphi}{\cosh\eta - \cos\xi} \\ Z = \dfrac{a\sinh\eta}{\cosh\eta - \cos\xi} \end{cases}.$$

   Find the basis vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ for the bi-spherical coordinate system in terms of the Cartesian unit vectors. Also find the scale factors, unit vectors, metric tensors (covariant and contravariant), Jacobian, volume element, and Christoffel symbols of the 1st kind.

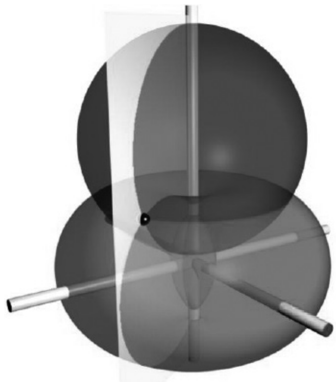


FIGURE 17.1 Bi-spherical coordinate system. <GNU Free Documentation License>

3. By rotating bipolar coordinates about Y-axis (the axis perpendicular to the line connecting the foci) we obtain a toroidal coordinate system, $\left(x^1,x^2,x^3\right)\equiv(\xi,\eta,\varphi)$ given in terms of Cartesian $\left(y^1,y^2,y^3\right)\equiv(X,Y,Z)$, as $\begin{cases} X=\dfrac{a\sinh\eta\cos\varphi}{\cosh\eta-\cos\xi} \\ Y=\dfrac{a\sinh\eta\sin\varphi}{\cosh\eta-\cos\xi} \\ Z=\dfrac{a\sin\xi}{\cosh\eta-\cos\xi} \end{cases}$. Find the basis vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ for the toroidal coordinate system in terms of the Cartesian unit vectors. Also find the scale factors, unit vectors, metric tensors (covariant and contravariant), Jacobian, volume element, and Christoffel symbols of the 2nd kind.

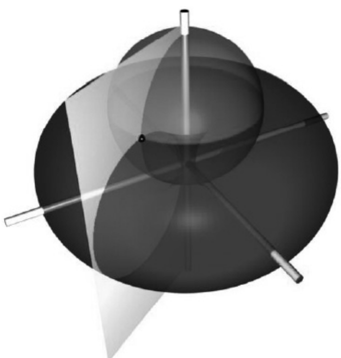


FIGURE 17.2 Toroidal coordinate system. <GNU Free Documentation License>

4. Having displacement vector $\vec{r} = y^i \vec{E}_i$ defined in Cartesian coordinate system y^i, express $\vec{r}$ in terms of the contravariant and physical components in cylindrical, spherical, bi-polar, bi-spherical, and toroidal coordinate systems. Use the definitions of these systems as given in previous examples and exercises.

5. Show that for an orthogonal coordinate system the relation $g_{ii} = 1/g^{ii} = h_i^2$ holds, i.e., the diagonal elements of covariant and contravariant metric tensors. Note, no sum on index i.

6. For an orthogonal coordinate system, show that the following relations holds for Christoffel symbols of the 2nd kind (all indices are set to be different):

$$\Gamma^k_{ij} = 0,\ \Gamma^j_{ii} = -\frac{h_i}{h_j^2}\frac{\partial h_i}{\partial x^j},\ \Gamma^i_{ij} = \frac{1}{h_i}\frac{\partial h_i}{\partial x^j},\ \Gamma^i_{ii} = \frac{1}{h_i}\frac{\partial h_i}{\partial x^i}.$$

7. Find the relations for Christoffel symbols of the 1st kind for cylindrical and spherical coordinate systems. Use results from the example given in Section 10.4.1.

8. Show that covariant differentiation obeys the same rule as does ordinary differentiation, when operating on products, e.g., $\left(A^i B^j\right)_{,k} = A^i_{,k} B^j + A^i B^j_{,k}$.

9. Write down the expression for the covariant derivative of a tensor of rank five, A^{ij}_{km}, using hints from Section 10.14.

10. Show that metric tensors behave like a constant under a covariant differentiation operation, i.e., $\left(A^i g_{ij}\right)_{,k} = A^i_{,k} g_{ij}$ and $\left(A_i g^{ij}\right)_{,k} = A_{i,k} g^{ij}$.

11. Show that the recursive relation for the curl of a vector $\vec{A}$ reads $\vec{A}(n) = -\nabla^2 \vec{A}(n-2)$, for $n > 2$. where n is the number of curl operations and $\vec{A}(n) = \underbrace{\vec{\nabla}\times\vec{\nabla}\times\cdots\times\vec{\nabla}}_{n\,times}\times\vec{A}$ (note, $\vec{A}(0) = \vec{A}$).

12. Using the recursive relation for the curl of a vector given in Exercise 11, show that the recursive relation for the biharmonic operator reads $\vec{A}(n) = \nabla^4 \vec{A}(n-4)$, for $n > 4$.

13. Using the results of Exercises 11 and 12, show that the recursive relations for higher order operator ∇^{2n} reads $\vec{A}(2n+1) = (-1)^n \nabla^{2n} \vec{A}(1)$ and $\vec{A}(2n) = -\nabla^{2n-2} \vec{A}(2)$.

14. For the 4D spherical coordinate system from Example 15.15, calculate the Christoffel symbols:

$$\Gamma^{r}=\begin{bmatrix}\Gamma^{r}_{rr} & \Gamma^{r}_{r\psi} & \Gamma^{r}_{r\theta} & \Gamma^{r}_{r\varphi}\\ \cdots & \Gamma^{r}_{\psi\psi} & \Gamma^{r}_{\psi\theta} & \Gamma^{r}_{\psi\varphi}\\ \cdots & \cdots & \Gamma^{r}_{\theta\theta} & \Gamma^{r}_{\theta\varphi}\\ \cdots & \cdots & \cdots & \Gamma^{r}_{\varphi\varphi}\end{bmatrix},\ \Gamma^{\psi}=\begin{bmatrix}\Gamma^{\psi}_{rr} & \Gamma^{\psi}_{r\psi} & \Gamma^{\psi}_{r\theta} & \Gamma^{\psi}_{r\varphi}\\ \cdots & \Gamma^{\psi}_{\psi\psi} & \Gamma^{\psi}_{\psi\theta} & \Gamma^{\psi}_{\psi\varphi}\\ \cdots & \cdots & \Gamma^{\psi}_{\theta\theta} & \Gamma^{\psi}_{\theta\varphi}\\ \cdots & \cdots & \cdots & \Gamma^{\psi}_{\varphi\varphi}\end{bmatrix},$$

$$\Gamma^{\theta}=\begin{bmatrix}\Gamma^{\theta}_{rr} & \Gamma^{\theta}_{r\psi} & \Gamma^{\theta}_{r\theta} & \Gamma^{\theta}_{r\varphi}\\ \cdots & \Gamma^{\theta}_{\psi\psi} & \Gamma^{\theta}_{\psi\theta} & \Gamma^{\theta}_{\psi\varphi}\\ \cdots & \cdots & \Gamma^{\theta}_{\theta\theta} & \Gamma^{\theta}_{\theta\varphi}\\ \cdots & \cdots & \cdots & \Gamma^{\theta}_{\varphi\varphi}\end{bmatrix},\ \Gamma^{\varphi}=\begin{bmatrix}\Gamma^{\varphi}_{rr} & \Gamma^{\varphi}_{r\psi} & \Gamma^{\varphi}_{r\theta} & \Gamma^{\varphi}_{r\varphi}\\ \cdots & \Gamma^{\varphi}_{\psi\psi} & \Gamma^{\varphi}_{\psi\theta} & \Gamma^{\varphi}_{\psi\varphi}\\ \cdots & \cdots & \Gamma^{\varphi}_{\theta\theta} & \Gamma^{\varphi}_{\theta\varphi}\\ \cdots & \cdots & \cdots & \Gamma^{\varphi}_{\varphi\varphi}\end{bmatrix}.$$

15. Show that the following expressions simplify as given (see Example 15.16):

a. $\underbrace{\vec{\nabla}}_{\times}\underbrace{A}_{\times}\overbrace{\vec{\nabla}}^{\times}\overbrace{\vec{A}}^{\times}=A_{k',j}A^{kj}_{,\cdot}+\vec{A}\cdot\left(\nabla^{2}\vec{A}\right)-A_{k',j}A^{jk}_{,\cdot}-\vec{A}\cdot\left(\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)\right)$

b. $\underbrace{\vec{\nabla}}_{\times}\overbrace{A}^{\times}\overbrace{\vec{\nabla}}^{\times}\underbrace{\vec{\nabla}}_{\times}\Phi=\left(\vec{\nabla}\cdot\vec{A}\right)\left(\nabla^{2}\Phi\right)-A^{k}_{,j}\left(\nabla_{k}\Phi\right)^{j}_{,\cdot}$

c. $\underbrace{\vec{\nabla}}_{\times}\overbrace{\vec{\nabla}}^{\times}\overbrace{A}^{\times}\underbrace{\vec{\nabla}}_{\times}\Phi=\nabla^{2}\left(\vec{A}\cdot(\nabla\Phi)\right)-\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)\cdot(\nabla\Phi)-A^{jk}_{,\cdot}\left(\nabla_{k}\Phi\right)-\left(\vec{\nabla}\cdot\vec{A}\right)$
$\left(\nabla^{2}\Phi\right)-\vec{A}\cdot\left(\vec{\nabla}\left(\nabla^{2}\Phi\right)\right)$

16. Write the below expressions in index notation:

a. $\left(\vec{\nabla}\vec{\nabla}\right)\cdot\vec{A}+\left(\vec{\nabla}\cdot\vec{V}\right)\vec{A}$

b. $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)$

c. $\vec{A}\cdot\left(\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)\right)$

17. Show that the following relations holds:

a. $\varepsilon^{ijk}\varepsilon_{lmn}=\begin{vmatrix}\delta^{i}_{l} & \delta^{i}_{m} & \delta^{i}_{n}\\ \delta^{j}_{l} & \delta^{j}_{m} & \delta^{j}_{n}\\ \delta^{k}_{l} & \delta^{k}_{m} & \delta^{k}_{n}\end{vmatrix}$, **b.** $\varepsilon^{ijk}\varepsilon_{imn}=\delta^{j}_{m}\delta^{k}_{n}-\delta^{j}_{n}\delta^{k}_{m}$, **c.** $\varepsilon^{ijk}\varepsilon_{ijn}=2\delta^{k}_{n}$

d. $\varepsilon^{ijk}\varepsilon_{ijk}=6$

18. Using Equation 9.13, write the determinant for a 5 × 5 matrix.

19. Use Equation 13.19 to find physical components of the convective time derivative of a velocity vector, and express the results for (3D) (hint: see Example 15.14):

a. Cartesian coordinates

b. Cylindrical coordinates

c. Spherical coordinates

20. Show that the non-zero Christoffel symbols of the 1st kind for cylindrical and spherical coordinates read:

a. Cylindrical coordinates: $\Gamma_{r\theta,\theta} = r$ and $\Gamma_{\theta\theta,r} = -r$.

b. Spherical coordinates: $\Gamma_{\varphi\varphi,r} = -r$, $\Gamma_{\theta\theta,r} = -r\sin^2\varphi$, $\Gamma_{\theta\theta,\varphi} = -\dfrac{r^2}{2}\sin 2\varphi$,

$\Gamma_{r\varphi,\varphi} = r$ and $\Gamma_{r\theta,\theta} = r\sin^2\varphi$, $\Gamma_{\varphi\theta,\theta} = \dfrac{r^2}{2}\sin 2\varphi$

21. Using cross-product expression for two vectors, show that ($\mathcal{J}$ is the Jacobian):

a. $\mathcal{J}\mathcal{E}_{ijk} = \dfrac{1}{\mathcal{J}}\mathcal{E}^{mnl} g_{mi} g_{nj} g_{lk}$

b. $\mathcal{J}^2 = \dfrac{1}{6}\mathcal{E}^{ijk}\mathcal{E}^{mnl} g_{mi} g_{nj} g_{kl}$

22. Calculate the single equivalent rotation angle and rotation axis for transforming Cartesian system (y_1, y_2, y_3) to (y'_1, y'_2, y'_3) such that y'_1 coincides with y_1, y'_2 coincides with y_3, and y'_3 coincides with y_1.

23. Find the rotation matrix for transforming (y_1, y_2, y_3):

a. Rotate about (0, 0, 1) for 45°

b. Then, rotate about (1, 0, 0) for 60°

c. Then, rotate about (0, 1, 0) for 75°

d. Find the final rotation matrix and equivalent rotation angle

e. Find equivalent axis of rotation

24. Show the following relations hold for unit quaternion $\hat{Q} = Q_0 + \vec{Q}$ and arbitrary vector $\vec{V}$.

a. $\hat{Q}\vec{V}\hat{Q}^* = \left(Q_0^2 - \left|\vec{Q}\right|^2\right)\vec{V} + 2\left(\vec{Q}\cdot\vec{V}\right)\vec{Q} + 2Q_0\left(\vec{Q}\times\vec{V}\right)$, active rotation

b. $\hat{Q}^*\vec{V}\hat{Q} = \left(2Q_0^2 - 1\right)\vec{V} + 2\left(\vec{V}\cdot\vec{Q}\right)\vec{Q} + 2Q_0\left(\vec{V}\times\vec{Q}\right)$, passive rotation

25. Derive The rotation matrix given by Equation 15.22, using unit quaternion $\hat{Q}$ and arbitrary vector $\vec{V}$.

26. Using Example 16.18 rotation matrix result, calculate the corresponding Euler angles.

27. Using Example 16.19 rotation matrix result, calculate the corresponding Euler angles.

28. Using the quaternions method as shown in Example 16.20, calculate the components of vector $\overrightarrow{OA} = (1,-1,2)$ after two successive rotations. The first rotation is by an angle of 80° about the unit vector $\vec{n} = \left(0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and the second rotation is by an angle of 50° about the unit vector $\vec{n} = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right)$. Perform the calculations for both active and passive rotation scenarios.

29. For Exercise 28, calculate the corresponding rotation matrix using the product of the quaternions involved. Also using the resulted rotation matrix calculate the equivalent axis and angles of single rotation.

30. A pilot maneuvers its airplane through a pitch-roll-yaw series or rotations. Using the relation for the corresponding rotation matrix $\mathbb{R}_{yxz}$ analyze the gimbal lock orientation when roll angle is equal to $\frac{3\pi}{2}$.

31. Show that quaternion operation for active rotation $\left(QVQ^*\right)$ preserves dot-products and triple products, hence it is a rotation.

32. Repeat the Exercise 16.20 considering the rotation as a passive one (i.e., coordinates are rotated).

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